

## 1 Engel's theorem

We briefly review the main items from the last lecture.

Let  $L$  be a Lie algebra over a field  $\mathbb{F}$ .

If  $I, J \subseteq L$  then define  $[I, J] = \mathbb{F}\text{-span}\{[X, Y] : X \in I \text{ and } Y \in J\}$ .

Some definitions from last time:

- $L$  is *solvable* if  $L^{(n)} = 0$  for some  $n \gg 0$  where  $L^{(0)} = L$  and  $L^{(n+1)} = [L^{(n)}, L^{(n)}]$ .
- $L$  is *nilpotent* if  $L^n = 0$  for some  $n \gg 0$  where  $L^0 = L$  and  $L^{n+1} = [L, L^n] = [L^n, L]$ .
- $L$  is *semisimple* if its unique maximal solvable ideal  $\text{Rad}(L)$  is zero.

We will only discuss semisimple Lie algebras that are finite-dimensional.

Our proof of the existence of the unique maximal solvable ideal  $\text{Rad}(L)$  required  $\dim(L) < \infty$ .

A linear map  $f : V \rightarrow V$  is *nilpotent* if  $f^n = 0$  for some  $n \gg 0$ .

An element  $X \in L$  is *ad-nilpotent* if the linear map  $\text{ad}_X = [X, \cdot] : L \rightarrow L$  is nilpotent.

Easy-to-prove fact: if  $L$  is a nilpotent Lie algebra then every  $X \in L$  is ad-nilpotent.

Harder-to-prove converse:

**Theorem** (Engel's theorem). If  $\dim(L) < \infty$  and every  $X \in L$  is ad-nilpotent then  $L$  is nilpotent.

One more result from last time, proved on the way to Engel's theorem.

Suppose  $V$  is a finite-dimensional  $\mathbb{F}$ -vector space and  $L \subseteq \mathfrak{gl}(V)$  is a Lie subalgebra.

Recall that if  $v_1, v_2, \dots, v_n$  is a basis for  $V$  and  $X \in \mathfrak{gl}(V)$  then the corresponding matrix of  $X$  is the square array  $[X_{ij}]_{1 \leq i, j \leq n}$  where  $X_{ij} \in \mathbb{F}$  is such that  $Xv_j = \sum_{i=1}^n X_{ij}v_i$ .

**Theorem.** If all elements of  $L$  are nilpotent linear maps  $V \rightarrow V$  then there exists a basis of  $V$  relative to which the matrices of all elements of  $L$  are strictly upper-triangular.

## 2 Lie's theorem

We now assume  $\mathbb{F}$  is an algebraically closed field of characteristic zero (like the complex numbers  $\mathbb{C}$ ).

We also assume that  $L$  is a Lie algebra over  $\mathbb{F}$  of finite dimension.

**Theorem.** Assume  $L$  is a solvable Lie subalgebra of  $\mathfrak{gl}(V)$  for some vector space  $V$  with  $0 < \dim(V) < \infty$ .

Then there exists a nonzero vector  $0 \neq v \in V$  that is an eigenvector for every  $X \in L$ , meaning that

$$Xv = \lambda(X)v \quad \text{for all } X \in L$$

for some linear map  $\lambda : X \rightarrow \mathbb{F}$ .

*Proof.* One can prove this result by the following steps, similar to the proof of the previous theorem:

- (0) Since  $\mathbb{F}$  is algebraically closed, any given  $X \in L$  has an eigenvector in  $V$ .

Hence the theorem holds if  $\dim(L) \leq 1$ . Assume  $\dim(L) > 1$ .

(1) Find an ideal  $K \subseteq L$  with  $\dim(L) = \dim(K) + 1$ .

This can be done by letting  $K$  be the preimage in  $L$  of any codimension one subspace of  $L/[L, L]$ .

This works since  $L/[L, L]$  is nonzero (as  $L$  is solvable) and abelian (by the definition of  $[L, L]$ ).

(2) By induction  $K$  has at least one common eigenvector in  $V$ , say with eigenvalue function  $\lambda : K \rightarrow \mathbb{F}$ .

(3) Let  $W = \{v \in V : Yv = \lambda(Y)v \text{ for all } Y \in K\}$  be the common  $\lambda$ -eigenspace for  $K$  in  $V$ .

Then we can check that  $L$  stabilizes this nonzero subspace. This is the hard part of the argument.

(4) Finally choose any  $Z \in L$  with  $Z \notin K$ . Step (3) shows that  $Z$  restricts to a linear map  $W \rightarrow W$ .

Since  $\mathbb{F}$  is algebraically closed,  $Z$  has an eigenvector  $v \in W$ .

Since  $\dim(L) = \dim(K) + 1$  we have  $L = K \oplus \mathbb{F}Z$  and  $v$  is a common eigenvector for all of  $L$ .

The eigenvalue map  $L \rightarrow \mathbb{F}$  is the linear extension of  $\lambda : K \rightarrow \mathbb{F}$  sending  $Z$  to its eigenvalue for  $v$ .

Apart from some minor details (see Humphrey's textbook) only step (3) requires more explanation. This is where we use the  $\text{char}(\mathbb{F}) = 0$  hypothesis.

We need to show that if  $X \in L$  and  $w \in W$  then  $Xw \in W$ , meaning that  $YXw = \lambda(Y)Xw$  for all  $Y \in K$ . By the Jacobi identity we know that

$$YXw = XYw - [X, Y]w = \lambda(Y)Xw - \lambda([X, Y])w.$$

So it is enough to check that  $\lambda([X, Y]) = 0$  for all  $Y \in K$ .

*Proof of this claim.* Let  $n > 0$  be minimal such that  $w, Xw, X^2w, \dots, X^nw$  are linearly dependent.

Define  $W_i = \mathbb{F}\text{-span}\{w, Xw, X^2w, \dots, X^{i-1}w\}$ . Then  $W_0 = 0$  and  $\dim(W_i) = \min\{i, n\}$ .

We argue that  $YX^i w \in \lambda(Y)X^i w + W_i$  for all  $Y \in K$ . This is clear if  $i = 0$  and if  $i > 0$  then

$$YX^i w = YXX^{i-1}w = X(YX^{i-1}w) - [X, Y]X^{i-1}w.$$

This expression is in  $\lambda(Y)X^i w + W_i$  when  $Y \in K$ , since then  $[X, Y] \in K$  so by induction

$$YX^{i-1}w \in \lambda(Y)X^{i-1}w + W_{i-1} \quad \text{and} \quad [X, Y]X^{i-1}w \in \lambda([X, Y])X^{i-1}w + W_{i-1} \subseteq W_i.$$

Thus, relative to the basis  $w, Xw, X^2w, \dots, X^{n-1}w$  each  $Y \in K$  operates on  $W_n$  as an upper- $\Delta$  matrix

$$\begin{bmatrix} \lambda(Y) & & * \\ & \ddots & \\ 0 & & \lambda(Y) \end{bmatrix}.$$

Therefore if  $Y \in K$  then  $\text{trace}(Y|_{W_n}) = n\lambda(Y)$  and also  $\text{trace}([X, Y]|_{W_n}) = n\lambda([X, Y])$  as  $[X, Y] \in K$ .

The notation  $f|_{W_n}$  here means the linear map  $W_n \rightarrow W_n$  obtained by restricting  $f$ .

But  $X$  and  $Y \in K$  both preserve  $W_n$  so  $\text{trace}([X, Y]|_{W_n}) = \text{trace}(X|_{W_n}Y|_{W_n}) - \text{trace}(Y|_{W_n}X|_{W_n}) = 0$ .

Thus  $n\lambda([X, Y]) = 0$  so since  $\text{char}(\mathbb{F}) = 0$  we conclude that  $\lambda([X, Y]) = 0$ . ■

Having checked this claim, we conclude that step (3) and the rest of the argument work as expected. □

**Theorem** (Lie’s theorem). Suppose  $L \subseteq \mathfrak{gl}(V)$  is a solvable Lie subalgebra where  $\dim(V) = n < \infty$ . Then there is some basis of  $V$  relative to which the matrices of all elements of  $L$  are upper-triangular.

*Proof.* Choose  $0 \neq v_1 \in V$  with  $Xv_1 = \lambda(X)v_1$  for all  $X \in L$  for some linear map  $\lambda : L \rightarrow \mathbb{F}$ .

Set  $V_1 = \mathbb{F}v_1$  and consider the quotient space  $V/V_1$ .

Apply the theorem inductively to obtain a basis  $v_2 + V_1, v_3 + V_1, \dots, v_n + V_1$  for  $V/V_1$  where each  $v_i \in V$ .

Then the desired basis for  $V$  is then  $v_1, v_2, v_3, \dots, v_n$ . □

**Corollary.** Suppose  $L$  is a solvable Lie algebra with  $\dim(L) = n < \infty$ .

Then there exists a chain of ideals  $0 = L_0 \subseteq L_1 \subseteq L_2 \subseteq \dots \subsetneq L_n = L$  with  $\dim(L_i) = i$ .

*Proof.* The Lie algebra  $\mathfrak{ad}(L) \subseteq \mathfrak{gl}(L)$  is a homomorphic image of  $L$  so is solvable.

Apply Lie’s theorem to this Lie algebra with  $V = L$ .

This gives a basis  $v_1, v_2, \dots, v_n \in L$  such that  $\mathfrak{ad}_X(v_i) \in \mathbb{F}\text{-span}\{v_1, v_2, \dots, v_i\}$  for all  $X \in L$ .

The subspaces  $L_i = \mathbb{F}\text{-span}\{v_1, v_2, \dots, v_i\}$  are therefore ideals, and they form the desired chain. □

We can now generalize the fact that  $\mathfrak{t}_n(\mathbb{F})$  is solvable and  $\mathfrak{n}_n(\mathbb{F}) = [\mathfrak{t}_n(\mathbb{F}), \mathfrak{t}_n(\mathbb{F})]$  is nilpotent.

**Corollary.** Suppose  $L$  is a solvable Lie algebra of finite dimension. Then  $[L, L]$  is nilpotent.

*Proof.* Choose a basis of  $L$  such that the matrices of  $\mathfrak{ad}_X \in \mathfrak{gl}(L)$  are all upper-triangular.

Then the matrix of  $\mathfrak{ad}_{[X,Y]} = [\mathfrak{ad}_X, \mathfrak{ad}_Y]$  is always strictly upper-triangular.

Hence  $\mathfrak{ad}_Z$  is nilpotent for all  $Z \in [L, L]$ , so Engel’s theorem implies that  $[L, L]$  is nilpotent. □

### 3 Remarks

Under the hypotheses that

- $\mathbb{F}$  is an algebraically closed field with  $\text{char}(\mathbb{F}) = 0$ , and
- $L \subseteq \mathfrak{gl}(V)$  for an  $\mathbb{F}$ -vector space  $V$  with  $\dim(V) = n < \infty$ ,

we have shown the following properties:

- (1) If  $L$  is solvable that there exists a basis of  $V$  that makes all elements of  $L$  upper-triangular.

Equivalently, if  $L$  is solvable then there is an injective morphism  $\iota : L \rightarrow \mathfrak{t}_n(\mathbb{F})$ .

- (2) If  $L$  is nilpotent, then  $L$  is also solvable.

But in this case  $L$  does not have to be strictly upper-triangular in the basis from (1).

In other words, we might have  $\iota(L) \not\subseteq \mathfrak{n}_n(\mathbb{F})$ .

**Example:** take  $V = \mathbb{F}^n$  and let  $L = \mathfrak{d}_n(\mathbb{F})$  be the abelian Lie algebra of diagonal matrices.

- (3) The vector space  $V$  must have a basis that makes all elements of  $L$  strictly upper-triangular (meaning that there is an injective morphism  $L \rightarrow \mathfrak{n}_n(\mathbb{F})$ ) when all elements  $L$  are nilpotent.

However,  $L$  can be a nilpotent Lie algebra without having this property.

**Example:** again let  $V = \mathbb{F}^n$  and  $L = \mathfrak{d}_n(\mathbb{F})$ .

## 4 Jordan decomposition

In this section we do not require  $\text{char}(\mathbb{F}) = 0$  but still assume  $\mathbb{F}$  is algebraically closed.

Let  $V$  be a finite-dimensional  $\mathbb{F}$ -vector space.

We say that  $X \in \mathfrak{gl}(V)$  is *semisimple* if  $X$  is *diagonalizable*, meaning  $V$  has a basis of eigenvectors for  $X$ .

Some standard properties from linear algebra (which you may check as exercises):

- (a) If  $X, Y \in \mathfrak{gl}(V)$  are semisimple with  $XY = YX$  then  $aX + bY$  is semisimple for all  $a, b \in \mathbb{F}$ .
- (b) If  $X \in \mathfrak{gl}(V)$  is semisimple and  $X$  preserves a subspace  $W \subseteq V$  then  $X|_W \in \mathfrak{gl}(W)$  is semisimple.

We now quote some less obvious, but still standard results linear algebra.

We assume these facts as background and will not prove them ourselves.

Recall that  $X \in \mathfrak{gl}(V)$  is *nilpotent* if  $X^n = 0$  for some  $n \gg 0$ .

**Proposition.** For each  $X \in \mathfrak{gl}(V)$ , there are unique elements  $X_{\text{ss}}, X_{\text{nil}} \in \mathfrak{gl}(V)$  such that

$$X_{\text{ss}} \text{ is semisimple, } X_{\text{nil}} \text{ is nilpotent, } X_{\text{ss}}X_{\text{nil}} = X_{\text{nil}}X_{\text{ss}}, \text{ and } \boxed{X = X_{\text{ss}} + X_{\text{nil}}}.$$

The boxed formula is called the *Jordan decomposition* (or *Jordan–Chevalley decomposition*) of  $X$ .

Proof idea: if  $V = \mathbb{F}^n$  and the Jordan canonical form of the matrix of  $X$  has blocks

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & \ddots \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{bmatrix}$$

then the Jordan canonical forms of  $X_{\text{ss}}$  and  $X_{\text{nil}}$  are respectively obtained by replacing these blocks by

$$\begin{bmatrix} \lambda & 0 & & 0 \\ & \lambda & 0 & \\ & & \lambda & \ddots \\ & & & \ddots & 0 \\ 0 & & & & \lambda \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix}.$$

**Proposition.** Let  $X \in \mathfrak{gl}(V)$ . There are polynomials  $p(x), q(x) \in x\mathbb{F}[x]$  with  $p(0) = q(0) = 0$  such that

$$X_{\text{ss}} = p(X) \quad \text{and} \quad X_{\text{nil}} = q(X).$$

Consequently, the operators  $X_{\text{ss}}$  and  $X_{\text{nil}}$  commute with any  $Y \in \mathfrak{gl}(V)$  that has  $XY = YX$ .

**Corollary.** If  $A \subseteq B \subseteq V$  are subspaces and  $X \in \mathfrak{gl}(V)$  has  $XB \subseteq A$  then  $X_{\text{ss}}B \subseteq A$  and  $X_{\text{nil}}B \subseteq A$ .

We now state some Lie theoretic results.

Continue to let  $V$  be a finite-dimensional  $\mathbb{F}$ -vector space.

**Proposition.** If  $X \in \mathfrak{gl}(V)$  is nilpotent then so is  $\text{ad}_X \in \mathfrak{gl}(\mathfrak{gl}(V))$ .

*Proof.* If  $X^n = 0$  then  $(\text{ad}_X)^{2n} = 0$  since  $(\text{ad}_X)^{2n}(Y) \in \mathbb{F}\text{-span}\{X^i Y X^{2n-i} : 0 \leq i \leq 2n\}$ . □

**Proposition.** If  $X \in \mathfrak{gl}(V)$  is semisimple then so is  $\text{ad}_X \in \mathfrak{gl}(\mathfrak{gl}(V))$ .

*Proof.* Suppose  $v_1, v_2, \dots, v_n$  is a basis of  $V$  and  $a_1, a_2, \dots, a_n \in \mathbb{F}$  are such that  $Xv_i = a_i v_i$ .

Let  $e_{ij} \in \mathfrak{gl}(V)$  be the linear map with  $e_{ij}(v_k) = \begin{cases} v_i & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$

Then  $\{e_{ij} : 1 \leq i, j \leq n\}$  is a basis for  $\mathfrak{gl}(V)$  and  $\text{ad}_X e_{ij} = (a_i - a_j)e_{ij}$  since

$$(\text{ad}_X e_{ij})v_k = [X, e_{ij}]v_k = X \underbrace{e_{ij}v_k}_{\in \mathbb{F}v_i} - e_{ij}Xv_k = a_i e_{ij}v_k - a_k e_{ij}v_k = (a_i - a_j)e_{ij}v_k$$

where the last equality uses the fact that  $a_j e_{ij}v_k = a_k e_{ij}v_k$  since both sides are zero if  $j \neq k$ .

Thus the basis  $\{e_{ij}\}$  for  $\mathfrak{gl}(V)$  consists of eigenvectors for  $\text{ad}_X$  which is therefore diagonalizable. □

**Proposition.** Suppose  $X \in \mathfrak{gl}(V)$  has Jordan decomposition  $X = X_{\text{ss}} + X_{\text{nil}}$ .

Then the Jordan decomposition of  $\text{ad}_X \in \mathfrak{gl}(\mathfrak{gl}(V))$  is  $\text{ad}_X = \text{ad}_{X_{\text{ss}}} + \text{ad}_{X_{\text{nil}}}$ .

In other words, we have  $(\text{ad}_X)_{\text{ss}} = \text{ad}_{X_{\text{ss}}}$  and  $(\text{ad}_X)_{\text{nil}} = \text{ad}_{X_{\text{nil}}}$ .

*Proof.* We have already seen that  $\text{ad}_{X_{\text{ss}}}$  is semisimple and  $\text{ad}_{X_{\text{nil}}}$  is nilpotent.

The adjoint representation is linear so  $\text{ad}_{X_{\text{ss}}} + \text{ad}_{X_{\text{nil}}} = \text{ad}_{X_{\text{ss}} + X_{\text{nil}}} = \text{ad}_X$ .

Finally, since  $X_{\text{ss}}$  and  $X_{\text{nil}}$  commute we have  $[X_{\text{ss}}, X_{\text{nil}}] = 0$ .

Therefore  $\text{ad}_{X_{\text{ss}}}$  and  $\text{ad}_{X_{\text{nil}}}$  also commute since  $[\text{ad}_{X_{\text{ss}}}, \text{ad}_{X_{\text{nil}}}] = \text{ad}_{[X_{\text{ss}}, X_{\text{nil}}]} = 0$  by the Jacobi identity.

Thus  $\text{ad}_X = \text{ad}_{X_{\text{ss}}} + \text{ad}_{X_{\text{nil}}}$  has the defining properties of the unique Jordan decomposition of  $\text{ad}_X$ . □