1 Review: Jordan decomposition

Let $\mathbb{F}$ be an algebraically closed field, not necessarily with $\mathrm{char}(\mathbb{F}) = 0$.

Let V be a finite-dimensional $\mathbb{F}$-vector space.

A linear map $X \in \mathfrak{gl}(V)$ is *nilpotent* if $X^n = 0$ for some $n \gg 0$.

A linear map $X \in \mathfrak{gl}(V)$ is *semisimple* if it is *diagonalizable*.

Linear combinations of commuting semisimple operators are semisimple.

The restriction of a semisimple operator to any subspace that it preserves is also semisimple.

For each $X \in \mathfrak{gl}(V)$, there are unique elements $X_{\mathrm{ss}}, X_{\mathrm{nil}} \in \mathfrak{gl}(V)$ such that

$$X_{\mathrm{ss}} \text{ is semisimple}, \quad X_{\mathrm{nil}} \text{ is nilpotent}, \quad X_{\mathrm{ss}}X_{\mathrm{nil}} = X_{\mathrm{nil}}X_{\mathrm{ss}}, \quad \text{and} \quad \boxed{X = X_{\mathrm{ss}} + X_{\mathrm{nil}}}.$$

The boxed formula is called the *Jordan decomposition* of X.

In fact, both X_{ss} and X_{nil} are elements of $\mathbb{F}\text{-span}\{X, X^2, X^3, \dots\}$.

If $X \in \mathfrak{gl}(V)$ is semisimple then $X = X_{\mathrm{ss}}$ and if $X \in \mathfrak{gl}(V)$ is nilpotent then $X = X_{\mathrm{nil}}$.

If $X \in \mathfrak{gl}(V)$ has Jordan decomposition $X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$ then $(\mathsf{ad}_X)_{\mathrm{ss}} = \mathsf{ad}_{X_{\mathrm{ss}}}$ and $(\mathsf{ad}_X)_{\mathrm{nil}} = \mathsf{ad}_{X_{\mathrm{nil}}}$.

Hence if $X \in \mathfrak{gl}(V)$ is semisimple (respectively, nilpotent) then so is $\mathsf{ad}_X \in \mathfrak{gl}(\mathfrak{gl}(V))$.

2 Cartan's criterion for solvability

We now assume (more strongly than above) that $\mathbb{F}$ is an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$.

Let V be an $\mathbb{F}$-vector space with $\dim(V) < \infty$

Our first goal today is to prove a simple condition that implies that a Lie subalgebra of $\mathfrak{gl}(V)$ is solvable.

Lemma. Let $A \subseteq B$ be two subspaces of $\mathfrak{gl}(V)$. Define

$$M = \{Y \in \mathfrak{gl}(V) : [Y, B] \subseteq A\}$$

and suppose $X \in M$ has $\mathsf{trace}(XY) = 0$ for all $Y \in M$. Then X is nilpotent.

Proof. One way to prove the lemma is by the following clever argument.

Write the Jordan decomposition of X as $X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$.

Let $v_1, v_2, \dots, v_n$ be a basis for V in which the matrix of X is in Jordan canonical form.

Then the matrices of X_{ss} and X_{nil} in this basis are respectively diagonal and strictly upper-triangular.

In particular we can write $X_{\mathrm{ss}}v_i = a_i v_i$ for some eigenvalues $a_i \in \mathbb{F}$.

Because $\mathrm{char}(\mathbb{F}) = 0$, we can define $E = \mathbb{Q}\text{-span}\{a_1, a_2, \dots, a_n\} \subseteq \mathbb{F}$.

We want to show that $X_{\mathrm{ss}} = 0$ as then $X = X_{\mathrm{nil}}$ is nilpotent.

This holds if and only if $a_1 = a_2 = \cdots = a_n = 0$, meaning that $E = 0$.

Let E^* be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-linear maps $E \to \mathbb{Q}$.

Since $\dim_{\mathbb{Q}}(E) \leq n < \infty$ it holds that $\dim_{\mathbb{Q}}(E^*) = \dim_{\mathbb{Q}}(E)$ and it suffices to show that $E^* = 0$.

Suppose $f \in E^*$. To prove that $E^* = 0$ we will show that $f = 0$.
Let $Y \in \mathfrak{gl}(V)$ be the linear map with $Yv_i = f(a_i)v_i$ for all i.
Let $e_{ij} \in \mathfrak{gl}(V)$ be the linear map that sends $v_k \mapsto \delta_{jk}v_i$ where $\delta_{jk} = |\{j\} \cap \{k\}|$.
As in the last lecture, one can check that $\mathsf{ad}_{X_{\mathrm{ss}}} e_{ij} = (a_i - a_j)e_{ij}$ and $\mathsf{ad}_Y e_{ij} = (f(a_i) - f(a_j))e_{ij}$.

It is a standard result in polynomial interpolation that some $r(x) \in \mathbb{Q}[x]$ exists with

$$r(a_i - a_j) = f(a_i) - f(a_j) = f(a_i - a_j) \qquad \text{for all } 1 \leq i, j \leq n.$$

(Look up *Lagrange polynomials* to see an explicit construction.)
Since $r(0) = r(a_i - a_i) = f(a_i - a_i) = f(0) = 0$ we have $r(x) \in x\mathbb{Q}[x]$.

We have $\mathsf{ad}_Y = r(\mathsf{ad}_{X_{\mathrm{ss}}})$ since both sides give the same result when applied to each e_{ij}.
Since $\mathsf{ad}_{X_{\mathrm{ss}}} = (\mathsf{ad}_X)_{\mathrm{ss}}$ we also have $\mathsf{ad}_{X_{\mathrm{ss}}} = p(\mathsf{ad}_X)$ for some polynomial $p(x) \in x\mathbb{F}[x]$.
Since we assume $X \in M$ it follows that $\mathsf{ad}_X(B) \subseteq A$ so

$$[Y, B] = \mathsf{ad}_Y(B) = r(\mathsf{ad}_{X_{\mathrm{ss}}})(B) = r(p(\mathsf{ad}_X))(B) \subseteq A.$$

This implies that $Y \in M$. Therefore our hypotheses assert that $\mathsf{trace}(XY) = 0$.

But the matrices of X_{ss} and Y are diagonal in the basis $v_1, v_2, \dots, v_n$ while X_{nil} is strictly upper-triangular.
Hence the matrix of $X_{\mathrm{nil}}Y$ is also strictly upper triangular so

$$0 = \mathsf{trace}(XY) = \mathsf{trace}(X_{\mathrm{ss}}Y) + \mathsf{trace}(X_{\mathrm{nil}}Y) = \mathsf{trace}(X_{\mathrm{ss}}Y) = \sum_{i=1}^{n} a_i f(a_i) \in E.$$

Thus $0 = f(0) = f(\mathsf{trace}(XY)) = \sum_{i=1}^{n} f(a_i)^2 \in \mathbb{Q}$ which can only hold if $f = 0$, as needed. □

The next proposition is much easier to prove.

Proposition. If $X, Y, Z \in \mathfrak{gl}(V)$ then $\mathsf{trace}([X, Y]Z) = \mathsf{trace}(X[Y, Z])$.

Proof. The two traces expand to $\mathsf{trace}(XYZ) - \mathsf{trace}(YXZ)$ and $\mathsf{trace}(XYZ) - \mathsf{trace}(XZY)$.
The general identity $\mathsf{trace}(AB) = \mathsf{trace}(BA)$ implies that $\mathsf{trace}(YXZ) = \mathsf{trace}(XZY)$. □

The previous results lead to the following theorem.

Theorem (Cartan's criterion)**.** Let $L \subseteq \mathfrak{gl}(V)$ be a Lie algebra where $\dim(V) < \infty$.
Suppose $\mathsf{trace}(XY) = 0$ for all $X \in [L, L]$ and $Y \in L$. Then L is solvable.

Proof. To show that L is solvable it suffices to check that $[L, L]$ is nilpotent.

For this, will show that every $X \in [L, L]$ is a nilpotent element of $\mathfrak{gl}(V)$.

This will imply that every element of $[L, L]$ is ad-nilpotent.

Then Engel's theorem tells us that $[L, L]$ is nilpotent.

Let $M = \{Y \in \mathfrak{gl}(V) : [Y, L] \subseteq [L, L]\}$.

We have $[L, L] \subseteq L \subseteq M$ but M might be strictly larger than L.

If $X_1, X_2 \in L$ and $Y \in \mathfrak{gl}(V)$ are any elements then by the previous proposition

$$\mathsf{trace}([X_1, X_2]Y) = \mathsf{trace}(X_1[X_2, Y]) = \mathsf{trace}([X_2, Y]X_1) = -\mathsf{trace}([Y, X_2]X_1).$$

By definition, if $Y \in M$ then $[Y, X_2] \in [L, L]$ and then by hypothesis $\mathsf{trace}([Y, X_2]X_1) = 0$.

Hence if $X_1, X_2 \in L$ and $Y \in M$ then $\mathsf{trace}([X_1, X_2]Y) = 0$.

But if $X \in [L, L]$, then X is a linear combination of brackets $[X_1, X_2]$ so $\mathsf{trace}(XY) = 0$ for all $Y \in M$.

We conclude by the lemma (with $A = [L, L]$ and $B = L$) that every $X \in [L, L]$ is nilpotent, as needed. □

3 Killing form

Let L be a finite-dimensional Lie algebra.

Definition. The *Killing form* of L is the bilinear form $\mathcal{K} : L \times L \to L$ defined by

$$\mathcal{K}(X, Y) = \mathsf{trace}(\mathsf{ad}_X \mathsf{ad}_Y) \quad \text{for } X, Y \in L.$$

To compute $\mathcal{K}(X, Y)$, pick a basis for L and then write down the matrices of ad_X and ad_Y in this basis.

Then multiply the matrices of ad_X and ad_Y and sum the diagonal entries in the product

We mention two easy properties of the Killing form:

Proposition. The Killing form is *symmetric* in the sense that $\mathcal{K}(X, Y) = \mathcal{K}(Y, X)$ for all $X, Y \in L$.

The Killing form is also *associative* in the sense that $\mathcal{K}([X, Y], Y) = \mathcal{K}(X, [Y, Z])$ for all $X, Y, Z \in L$.

The second property is equivalent to a result proved earlier today.

Example. Let us compute the Killing form when $L = \mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}\text{-span}\{E, H, F\}$ for

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{and} \qquad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Check that $[E, E] = [H, H] = [F, F] = 0$ and $[E, H] = -2E$ and $[F, H] = 2F$ and $[E, F] = H$.

Relative to the ordered basis E, H, F, we therefore have

$$\mathsf{ad}_E = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathsf{ad}_H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \text{and} \qquad \mathsf{ad}_F = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

This lets us compute
$$\begin{bmatrix} \mathcal{K}(E,E) & \mathcal{K}(E,H) & \mathcal{K}(E,F) \\ \mathcal{K}(H,E) & \mathcal{K}(H,H) & \mathcal{K}(H,F) \\ \mathcal{K}(F,E) & \mathcal{K}(F,H) & \mathcal{K}(F,F) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

The *radical* of $\mathcal{K} : L \times L \to L$ (or more generally of any symmetric bilinear form) is the subspace
$$\mathsf{Rad}(\mathcal{K}) = \{X \in L : \mathcal{K}(X,Y) = 0 \text{ for all } Y \in L\} = \{Y \in L : \mathcal{K}(X,Y) = 0 \text{ for all } X \in L\}.$$

The form $\mathcal{K}$ is *non-degenerate* if $\mathsf{Rad}(\mathcal{K}) = 0$.

We have $\mathsf{Rad}(\mathcal{K}) = 0$ precisely when the map $\mathcal{K}(X,\cdot) : L \to L$ is zero if and only if $X = 0$.

Less obviously, if $X_1, X_2, \dots, X_n$ is any basis for L then:

Proposition. We have $\mathsf{Rad}(\mathcal{K}) = 0$ if and only if the $n \times n$ matrix $\left[\ \mathcal{K}(X_i, X_j)\ \right]_{1 \le i,j \le n}$ is invertible.

Proof. If M denotes this matrix and $Y = \sum_{i=1}^n a_i X_i$ and $Z = \sum_{i=1}^n b_i X_i$ for some $a_i, b_i \in \mathbb{F}$, then
$$\mathcal{K}(Y,Z) = a^\top M b \qquad \text{for the vectors } a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If M is invertible and $b \neq 0$ then $Mb \neq 0$ so $\mathcal{K}(Y,Z) \neq 0$ when we take $a = Mb$.

Thus if M is invertible then $\mathcal{K}(Y,Z)$ for all $Y \in L$ only when $Z = 0$, so $\mathsf{Rad}(\mathcal{K}) = 0$.

Conversely, if M is not invertible then some nonzero b has $Mb = 0$.

In this case the corresponding nonzero $Z \in L$ belongs to $\mathsf{Rad}(\mathcal{K})$. □

We see from this proposition that the Killing form for $\mathfrak{sl}_2(\mathbb{F})$ is non-degenerate if and only if $\mathrm{char}(\mathbb{F}) \neq 2$.

Proposition. Let $I \subseteq L$ be an ideal.

Then the Killing form $\mathcal{K}_I$ of I is the restriction of the Killing form $\mathcal{K} = \mathcal{K}_L$ of L.

Proof. If $\phi : V \to W \subseteq V$ is a linear map where W and V are finite-dimensional vector spaces, then
$$\mathsf{trace}(\phi) = \mathsf{trace}(\phi|_W)$$
since if $w_1, w_2, \dots, w_k$ is a basis for W and $w_{k+1}, w_{k+1} \dots, w_n$ extends this to a basis of V then the corresponding matrix of ϕ has the block diagonal form
$$\left[\begin{array}{c|c} A & * \\ \hline 0 & 0 \end{array}\right]$$
where A is the matrix of $\phi|_W$ in the basis $w_1, w_2, \dots, w_k$.

Now apply this observation with $V = L$ and $W = I$ and $\phi = \mathsf{ad}_X \mathsf{ad}_Y$. □

Recall that L is *semisimple* if it has no nonzero solvable ideals, or equivalently if $\mathsf{Rad}(L) = 0$.

Theorem. A finite-dimensional Lie algebra is semisimple if and only if its Killing form is non-degenerate.

Equivalently, when $\dim(L) < \infty$ we have $\mathsf{Rad}(L) = 0$ if and only if $\mathsf{Rad}(\mathcal{K}) = 0$.

Proof. Let $S = \mathsf{Rad}(\mathcal{K})$. The associativity of the Killing form implies that S is an ideal of L.

In fact, Cartan's criterion implies that $\mathsf{ad}(S)$ is a solvable ideal of $\mathfrak{gl}(L)$, since

$$\mathsf{trace}(\mathsf{ad}_X \mathsf{ad}_Y) = \mathcal{K}(X, Y) = 0 \quad \text{for all } X \in S \supseteq [S, S] \text{ and all } Y \in L \supseteq S.$$

The center $Z(L)$ is abelian and hence solvable, and contained in S, so $\mathsf{ad}(S) \cong S/Z(L)$.

Since $Z(L)$ and $S/Z(L)$ are both solvable, we know from an earlier lemma that S is also solvable.

Thus if the largest solvable ideal $\mathsf{Rad}(L) = 0$ is zero then we also have $S = 0$.

Now suppose conversely that $S = 0$. We want to show that $\mathsf{Rad}(L) = 0$.

Assume $I \subseteq L$ is an abelian ideal. It suffices to check that $I \subseteq S$ (which means that $I = 0$ when $S = 0$).

This is because if $\mathsf{Rad}(L)$ is nonzero then $\mathsf{Rad}(L)^{(n)}$ is a nonzero abelian ideal for some $n \geq 1$.

Let $X \in I$ and $Y \in L$. Then $\mathsf{ad}_X \mathsf{ad}_Y$ is a map $L \to I$ and $(\mathsf{ad}_X \mathsf{ad}_Y)^2$ is a map $L \to [I, I] = 0$.

Thus $\mathsf{ad}_X \mathsf{ad}_Y$ is nilpotent so its eigenvalues are all 0, so its trace (the sum of the eigenvalues) is also 0.

This shows that $\mathcal{K}(X, Y) = 0$ for all $X \in I$ and $Y \in L$, which means that $I \subseteq S$ as needed. □