1 Review: Cartan's criterion and the Killing form

Define all vector spaces and Lie algebras over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$.

Let V be a finite-dimensional $\mathbb{F}$-vector space and suppose $L \subseteq \mathfrak{gl}(V)$ is a Lie algebra.

Theorem (Cartan's criterion)**.** Suppose $\mathsf{trace}(XY) = 0$ for all $X \in [L, L]$ and $Y \in L$. Then L is solvable.

The *radical* $\mathsf{Rad}(L)$ of a finite-dimensional Lie algebra L is its maximal solvable ideal.

The *Killing form* of a finite-dimensional Lie algebra L is the bilinear form $\mathcal{K} : L \times L \to L$ defined by

$$\mathcal{K}(X, Y) = \mathsf{trace}(\mathsf{ad}_X \mathsf{ad}_Y) \quad \text{for } X, Y \in L.$$

This symmetric bilinear form has the associativity property $\mathcal{K}([X, Y], Z) = \mathcal{K}(X, [Y, Z])$.

The *radical* of $\mathcal{K}$ is $\mathsf{Rad}(\mathcal{K}) = \{X \in L : \mathcal{K}(X, Y) = 0 \text{ if } Y \in L\} = \{Y \in L : \mathcal{K}(X, Y) = 0 \text{ if } X \in L\}$.

The form $\mathcal{K}$ is *non-degenerate* if $\mathsf{Rad}(\mathcal{K}) = 0$.

Theorem. A finite-dimensional Lie algebra is semisimple if and only if its Killing form is non-degenerate.

Equivalently, when $\dim(L) < \infty$ we have $\mathsf{Rad}(L) = 0$ if and only if $\mathsf{Rad}(\mathcal{K}) = 0$.

2 Simple ideals

A Lie algebra L is a *direct sum* of ideals $I_1, I_2, \dots, I_n$ if $L = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ as vector spaces.

This means that each $X \in L$ has a unique expression as

$$X = X_1 + X_2 + \cdots + X_n$$

with $X_j \in I_j$. The uniqueness in such expressions means that $I_j \cap I_k = 0$ for all $j \neq k$.

Since each I_j is an ideal we must also have $[I_j, I_k] = 0$ for all $j \neq k$ as $[I_j, I_k] \subseteq I_j \cap I_k$.

Recall that a Lie algebra is *simple* if it is non-abelian with no nonzero proper ideals.

Continue to assume $\mathbb{F}$ is algebraically closed with $\mathrm{char}(\mathbb{F}) = 0$.

Let L be a finite-dimensional Lie algebra defined over $\mathbb{F}$.

Theorem. Suppose L is semisimple. Then there exist simple ideals $L_1, L_2, \dots, L_n \subseteq L$ such that

(1) $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, and

(2) any simple ideal of L is equal to some L_i.

Proof. Notice that L is non-abelian since $\mathsf{Rad}(L) = 0$.

Let I be a nonzero proper ideal of L.

Write $\mathcal{K}$ for the Killing form of L. We showed last time that the Killing form of I is the restriction of $\mathcal{K}$.

Define $I^\perp = \{X \in L : \mathcal{K}(X, Y) = 0 \text{ for all } Y \in I\}$.

We claim that (a) $I^\perp$ is an ideal and (b) $L = I \oplus I^\perp$.

If we can prove these claims then (1) will follow by induction on $\dim(L)$, since if L has no nonzero proper ideals then L is simple and since any simple ideal of I or $I^\perp$ is also a simple ideal of L since $[I, I^\perp] = 0$.

To check (a), let $X \in L$, $Y \in I^\perp$, and $Z \in I$.

Then $\mathcal{K}([X,Y],Z) = -\mathcal{K}([Y,X],Z) = -\mathcal{K}(Y,[X,Z]) = 0$ since $[X,Z] \in I$. Hence $[X,Y] \in I^\perp$.

To check (b), note that $Z(L) = 0$ since L is semisimple, so $\mathsf{ad} : L \to \mathfrak{gl}(L)$ is injective.

Cartan's criterion applied to $I \cap I^\perp \cong \mathsf{ad}(I \cap I^\perp) \subseteq \mathfrak{gl}(L)$ implies that $I \cap I^\perp$ solvable, since

$$\mathsf{trace}(\mathsf{ad}_X \mathsf{ad}_Y) = \mathcal{K}(X,Y) = 0 \quad \text{for all } X \in [I \cap I^\perp, I \cap I^\perp] \subseteq L \text{ and } Y \in I \cap I^\perp \subseteq I^\perp.$$

Hence $I \cap I^\perp = 0$ as L is semisimple and $I \cap I^\perp$ is a solvable ideal.

As the Killing form is non-degenerate, we have $\dim(L) = \dim(I) + \dim(I^\perp)$ so $L = I \oplus I^\perp$ as claimed.

This completes the proof of (1).

Next suppose I is a simple ideal of $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$.

It remains to show (2) that the nonzero ideal I is equal to L_i for some i.

To prove this, check that $[I, L]$ is also an ideal of L using the Jacobi identity.

We cannot have $[I, L] = 0$ as then $I \subseteq Z(L) = 0$. Therefore $I = [I, L]$ as I is simple.

But then $I = [I, L] = \bigoplus_{i=1}^n [I, L_i]$ is simple so we must have

$$I = [I, L_i] = L_i \quad \text{for some } i \qquad \text{and} \qquad [I, L_j] = 0 \quad \text{when } i \neq j.$$

□

Our original definition of *semisimple* was the property having no nonzero solvable ideals.

Now we have a more intuitive characterization:

Corollary. The Lie algebra L is semisimple if and only if it is a direct sum of simple Lie algebras.

Proof. The previous theorem shows the "only if" direction.

Conversely, suppose $L = L_1 \oplus \cdots \oplus L_n$ where each L_i is simple. If L has Killing form $\mathcal{K}$ then

$$\mathsf{Rad}(\mathcal{K}) = \bigoplus_{i=1}^n \mathsf{Rad}(\mathcal{K}|_{L_i \times L_i}) \qquad \text{since} \qquad L_i^\perp = \bigoplus_{j \neq i} L_j.$$

The restricted form $\mathcal{K}|_{L_i \times L_i}$ are the Killing form of L_i.

As each simple L_i is semisimple, we have $\mathsf{Rad}(\mathcal{K}|_{L_i \times L_i}) = 0$ so $\mathsf{Rad}(\mathcal{K}) = 0$ and L is semisimple. □

Corollary. If L is semisimple then so are all ideals and homomorphic images of L, and $L = [L, L]$.

Proof. Assume L is semisimple and $I \subseteq L$ is an ideal.

Then the argument in the proof of the theorem shows that I is a direct sum of simple ideals.

Therefore I is semisimple, and L/I is also a direct sum of simple ideals so is semsimple.

In particular, this implies that all homomorphic images of L are semisimple.

Since L is semsimple we can write $L = \bigoplus_{i=1}^n L_i$ where each L_i is simple.

Then $[L_i, L_i] = L_i$ and $[L_i, L_j] = 0$ for all $i \neq j$, so $[L, L] = \bigoplus_{i=1}^n \bigoplus_{j=1}^n [L_i, L_j] = \bigoplus_{i=1}^n L_i = L$. □

3 Representations of Lie algebras: basic terminology

For this section, we fix an arbitrary field $\mathbb{F}$ and define all vector spaces and Lie algebras relative to $\mathbb{F}$.

Suppose L is a Lie algebra.

An *L-representation* is a Lie algebra morphism $\phi : L \to \mathfrak{gl}(V)$ for some vector space V.

Explicitly, this means a linear map ϕ with $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in L$.

An *L-module* is a vector space V with a bilinear operation

$$\begin{array}{ccc} L \times V & \to & L \\ (X, v) & \mapsto & X \cdot v \end{array}$$

such that $[X, Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v)$ for all $X, Y \in L$ and $v \in V$.

L-representations and L-modules are equivalent notions, just using different syntax.

Any L-representation can be converted to an L-module:

Proposition. If $\phi : L \to \mathfrak{gl}(V)$ is an L-representation then V is an L-module for the action

$$X \cdot v \overset{\text{def}}{=} \phi(X)v \qquad \text{for } X \in L \text{ and } v \in V.$$

Proof. The module action is bilinear since ϕ is linear.

Let $v \in V$. Since for any $A, B \in \mathfrak{gl}(V)$ we have $A(Bv) = (AB)v$, it follows for $X, Y \in L$ that

$$X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = (\phi(X)\phi(Y) - \phi(Y)\phi(X))\, v = [\phi(X), \phi(Y)]v = \phi([X, Y])v = [X, Y] \cdot v$$

as needed. $\square$

Likewise, any L-module can be converted to an L-representation:

Proposition. If V is an L-module then the map

$$\phi : L \to \mathfrak{gl}(V)$$

defined by $\phi(X) : v \mapsto X \cdot v$ for $X \in L$ is an L-representation.

The proof is similar to the previous proposition.

The operations going between L-representations and L-modules are inverses of each other.

A *submodule* of an L-module V is a subspace $U \subseteq V$ such that $X \cdot u \in U$ for all $X \in L$ and $u \in U$.

Any submodule is itself an L-module.

A *morphism* $f : V \to W$ of two L-modules is a linear map such that

$$f(X \cdot v) = X \cdot f(v) \quad \text{for all } v \in V \text{ and } X \in L.$$

The *kernel* of such a morphism is the subspace

$$\ker(f) = \{v \in V : f(v) = 0\}.$$

This is a submodule since if $f(v) = 0$ then $f(X \cdot v) = X \cdot f(v) = X \cdot 0 = 0$ (as the $\cdot$ action is bilinear).

An L-module morphism $V \to W$ is an *isomorphism* if it is a bijection.

An L-module V is *irreducible* if only submodules are 0 and $V \neq 0$.

If V and W and L-modules then the vector space direct sum $V \oplus W$ is also an L-module for the action

$$X \cdot (v + w) = X \cdot v + X \cdot w.$$

An L-module V is *completely reducible* if there are irreducible L-submodules $V_i \subseteq V$ such that $V = \bigoplus_i V_i$.

The L-module 0 is not considered irreducible because we want the direct sum decomposition of a completely reducible L-module to be unique (up to isomorphism and rearrangement of irreducible factors).

A *scalar map* $V \to V$ is a linear map of the form $v \mapsto \lambda v$ for some fixed $\lambda \in \mathbb{F}$.

An L-representation is *irreducible* if the associated L-module structure on V is an irreducible module.

We state the following fundamental result without proof.

Theorem (Schur's lemma)**.** Assume $\mathbb{F}$ is algebraically closed with $\mathrm{char}(\mathbb{F}) = 0$.

Suppose $\phi : L \to \mathfrak{gl}(V)$ is an irreducible L-representation.

Then the only linear maps $f : V \to V$ with $f\phi(X) = \phi(X)f$ for all $X \in L$ are the scalar maps $V \to V$.

Continue to suppose V is an L-module. Define V^* to be the vector space of all linear maps $V \to \mathbb{F}$.

Proposition. The vector space V^* is an L-module for the action

$$X \cdot f = \Big(\text{the linear map } V \to \mathbb{F} \text{ sending } \boxed{v \mapsto -f(X \cdot v)} \Big) \qquad \text{for } X \in L \text{ and } f \in V^*.$$

Note the unexpected minus sign in this formula.

Proof. Checking that the L-action on V^* is bilinear is straightforward.

Let $X, Y \in L$ and $f \in V^*$. Then for any $v \in V$ we have

$$(X \cdot (Y \cdot f))(v) = -(Y \cdot f)(X \cdot v) = f(Y \cdot (X \cdot v)) \quad \text{and similarly} \quad (Y \cdot (X \cdot f))(v) = f(X \cdot (Y \cdot v)).$$

On the other hand

$$([X, Y] \cdot f)(v) = -f([X, Y] \cdot v) = -f(X \cdot (Y \cdot v) - Y \cdot (X \cdot v)) = f(Y \cdot (X \cdot v)) - f(X \cdot (Y \cdot v)).$$

Comparing formulas shows that $[X, Y] \cdot f = Y \cdot (X \cdot f) - Y \cdot (X \cdot f)$ as needed. $\square$

Suppose V and W are two L-modules, say with bases $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$.

The *tensor product* $V \otimes W$ is the vector space spanned by all *tensors* $v \otimes w$ with $v \in V$ and $w \in W$ where

$$(v + v') \otimes w = v \otimes w + v' \otimes w \quad \text{and} \quad v \otimes (w + w') = v \otimes w + v \otimes w' \quad \text{and} \quad (av) \otimes w = v \otimes (aw)$$

for $v, v' \in V$ and $w, w' \in W$ and $a \in \mathbb{F}$.

Nontrivial fact: the vector space $V \otimes W$ has a basis given by $\{v_i \otimes w_j\}_{(i,j) \in I \times J}$.

Proposition. The tensor product $V \otimes W$ is an L-module for the bilinear action satisfying

$$X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot w) \qquad \text{for } X \in L,\ v \in V, \text{ and } w \in W.$$

Proof. If $X, Y \in L$, $v \in V$, and $w \in W$ then we can expand

$$[X,Y] \cdot (v \otimes w) = X \cdot (Y \cdot v) \otimes w - Y \cdot (X \cdot v) \otimes w + v \otimes X \cdot Y(\cdot w) - v \otimes Y \cdot (X \cdot w)$$

while

$$X \cdot (Y \cdot (v \otimes w)) - Y \cdot (X \cdot (v \otimes w)) = X \cdot (Y \cdot v \otimes w + v \otimes Y \cdot w) - Y \cdot (X \cdot v \otimes w + v \otimes X \cdot w).$$

If we further expand the second expression and cancel some terms then we recover the first expression. □

We mention one final property today.

Proposition. Let V be a finite-dimensional vector space. Then the linear map $\Phi : V^* \otimes V \to \mathfrak{gl}(V)$ with

$$\Phi(f \otimes v) = \Big(\text{the linear map } V \to V \text{ sending } \boxed{x \mapsto f(x)v} \Big) \qquad \text{for } f \in V^* \text{ and } v \in V$$

is an isomorphism of vector spaces.

Proof. As $\dim(\mathfrak{gl}(V)) = \dim(V)^2 = \dim(V^*)\dim(V) = \dim(V^* \otimes V)$ we just need to show Φ is surjective.

We may assume $V = \mathbb{F}^n$ is the space of n-row column vectors.

Then we can think of V^* as the space of $1 \times n$ matrices and $\mathfrak{gl}(V)$ as the space of all $n \times n$ matrices.

In this setup Φ is just the out-of-order matrix multiplication map

$$\Phi : \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \mapsto \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \in \mathfrak{gl}(V).$$

By choosing $a_i, b_j \in \{0, 1\}$ appropriately the right side can give any elementary matrix $E_{ij} \in \mathfrak{gl}(V)$.

Hence by linearity every $n \times n$ matrix occurs in the image of Φ, so Φ is surjective. □

If V is a finite-dimensional L-module, then there is a unique L-module structure on $\mathfrak{gl}(V)$ that makes Φ into a module isomorphism. Working through the definitions shows that this module has the formula

$$X \cdot f = \Big(\text{the linear map } V \to V \text{ sending } \boxed{v \mapsto X \cdot f(v) - f(X \cdot v)} \Big) \qquad \text{for } X \in L \text{ and } f \in \mathfrak{gl}(V).$$