

## 1 Review: Cartan's criterion and the Killing form

Define all vector spaces and Lie algebras over an algebraically closed field  $\mathbb{F}$  with  $\text{char}(\mathbb{F}) = 0$ .

Let  $V$  be a finite-dimensional  $\mathbb{F}$ -vector space and suppose  $L \subseteq \mathfrak{gl}(V)$  is a Lie algebra.

**Theorem** (Cartan's criterion). Suppose  $\text{trace}(XY) = 0$  for all  $X \in [L, L]$  and  $Y \in L$ . Then  $L$  is solvable.

The *radical*  $\text{Rad}(L)$  of a finite-dimensional Lie algebra  $L$  is its maximal solvable ideal.

The *Killing form* of a finite-dimensional Lie algebra  $L$  is the bilinear form  $\mathcal{K} : L \times L \rightarrow \mathbb{F}$  defined by

$$\mathcal{K}(X, Y) = \text{trace}(\text{ad}_X \text{ad}_Y) \quad \text{for } X, Y \in L.$$

This symmetric bilinear form has the associativity property  $\mathcal{K}([X, Y], Z) = \mathcal{K}(X, [Y, Z])$ .

The *radical* of  $\mathcal{K}$  is  $\text{Rad}(\mathcal{K}) = \{X \in L : \mathcal{K}(X, Y) = 0 \text{ if } Y \in L\} = \{Y \in L : \mathcal{K}(X, Y) = 0 \text{ if } X \in L\}$ .

The form  $\mathcal{K}$  is *non-degenerate* if  $\text{Rad}(\mathcal{K}) = 0$ .

**Theorem.** A finite-dimensional Lie algebra is semisimple if and only if its Killing form is non-degenerate.

Equivalently, when  $\dim(L) < \infty$  we have  $\text{Rad}(L) = 0$  if and only if  $\text{Rad}(\mathcal{K}) = 0$ .

## 2 Simple ideals

A Lie algebra  $L$  is a *direct sum* of ideals  $I_1, I_2, \dots, I_n$  if  $L = I_1 \oplus I_2 \oplus \dots \oplus I_n$  as vector spaces.

This means that each  $X \in L$  has a unique expression as

$$X = X_1 + X_2 + \dots + X_n$$

with  $X_j \in I_j$ . The uniqueness in such expressions means that  $I_j \cap I_k = 0$  for all  $j \neq k$ .

Since each  $I_j$  is an ideal we must also have  $[I_j, I_k] = 0$  for all  $j \neq k$  as  $[I_j, I_k] \subseteq I_j \cap I_k$ .

Recall that a Lie algebra is *simple* if it is non-abelian with no nonzero proper ideals.

Continue to assume  $\mathbb{F}$  is algebraically closed with  $\text{char}(\mathbb{F}) = 0$ .

Let  $L$  be a finite-dimensional Lie algebra defined over  $\mathbb{F}$ .

**Theorem.** Suppose  $L$  is semisimple. Then there exist simple ideals  $L_1, L_2, \dots, L_n \subseteq L$  such that

- (1)  $L = L_1 \oplus L_2 \oplus \dots \oplus L_n$ , and
- (2) any simple ideal of  $L$  is equal to some  $L_i$ .

*Proof.* Notice that  $L$  is non-abelian since  $\text{Rad}(L) = 0$ .

Let  $I$  be a nonzero proper ideal of  $L$ .

Write  $\mathcal{K}$  for the Killing form of  $L$ . We showed last time that the Killing form of  $I$  is the restriction of  $\mathcal{K}$ .

Define  $I^\perp = \{X \in L : \mathcal{K}(X, Y) = 0 \text{ for all } Y \in I\}$ .

We claim that (a)  $I^\perp$  is an ideal and (b)  $L = I \oplus I^\perp$ .

If we can prove these claims then (1) will follow by induction on  $\dim(L)$ , since if  $L$  has no nonzero proper ideals then  $L$  is simple and since any simple ideal of  $I$  or  $I^\perp$  is also a simple ideal of  $L$  since  $[I, I^\perp] = 0$ .

To check (a), let  $X \in L$ ,  $Y \in I^\perp$ , and  $Z \in I$ .

Then  $\mathcal{K}([X, Y], Z) = -\mathcal{K}([Y, X], Z) = -\mathcal{K}(Y, [X, Z]) = 0$  since  $[X, Z] \in I$ . Hence  $[X, Y] \in I^\perp$ .

To check (b), note that  $Z(L) = 0$  since  $L$  is semisimple, so  $\text{ad} : L \rightarrow \mathfrak{gl}(L)$  is injective.

Cartan's criterion applied to  $I \cap I^\perp \cong \text{ad}(I \cap I^\perp) \subseteq \mathfrak{gl}(L)$  implies that  $I \cap I^\perp$  solvable, since

$$\text{trace}(\text{ad}_X \text{ad}_Y) = \mathcal{K}(X, Y) = 0 \quad \text{for all } X \in [I \cap I^\perp, I \cap I^\perp] \subseteq L \text{ and } Y \in I \cap I^\perp \subseteq I^\perp.$$

Hence  $I \cap I^\perp = 0$  as  $L$  is semisimple and  $I \cap I^\perp$  is a solvable ideal.

As the Killing form is non-degenerate, we have  $\dim(L) = \dim(I) + \dim(I^\perp)$  so  $L = I \oplus I^\perp$  as claimed.

This completes the proof of (1).

Next suppose  $I$  is a simple ideal of  $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ .

It remains to show (2) that the nonzero ideal  $I$  is equal to  $L_i$  for some  $i$ .

To prove this, check that  $[I, L]$  is also an ideal of  $L$  using the Jacobi identity.

We cannot have  $[I, L] = 0$  as then  $I \subseteq Z(L) = 0$ . Therefore  $I = [I, L]$  as  $I$  is simple.

But then  $I = [I, L] = \bigoplus_{i=1}^n [I, L_i]$  is simple so we must have

$$I = [I, L_i] = L_i \quad \text{for some } i \quad \text{and} \quad [I, L_j] = 0 \quad \text{when } i \neq j.$$

□

Our original definition of *semisimple* was the property having no nonzero solvable ideals.

Now we have a more intuitive characterization:

**Corollary.** The Lie algebra  $L$  is semisimple if and only if it is a direct sum of simple Lie algebras.

*Proof.* The previous theorem shows the “only if” direction.

Conversely, suppose  $L = L_1 \oplus \cdots \oplus L_n$  where each  $L_i$  is simple. If  $L$  has Killing form  $\mathcal{K}$  then

$$\text{Rad}(\mathcal{K}) = \bigoplus_{i=1}^n \text{Rad}(\mathcal{K}|_{L_i \times L_i}) \quad \text{since} \quad L_i^\perp = \bigoplus_{j \neq i} L_j.$$

The restricted form  $\mathcal{K}|_{L_i \times L_i}$  is the Killing form of  $L_i$ .

As each simple  $L_i$  is semisimple, we have  $\text{Rad}(\mathcal{K}|_{L_i \times L_i}) = 0$  so  $\text{Rad}(\mathcal{K}) = 0$  and  $L$  is semisimple. □

**Corollary.** If  $L$  is semisimple then so are all ideals and homomorphic images of  $L$ , and  $L = [L, L]$ .

*Proof.* Assume  $L$  is semisimple and  $I \subseteq L$  is an ideal.

Then the argument in the proof of the theorem shows that  $I$  is a direct sum of simple ideals.

Therefore  $I$  is semisimple, and  $L/I$  is also a direct sum of simple ideals so is semisimple.

In particular, this implies that all homomorphic images of  $L$  are semisimple.

Since  $L$  is semisimple we can write  $L = \bigoplus_{i=1}^n L_i$  where each  $L_i$  is simple.

Then  $[L_i, L_i] = L_i$  and  $[L_i, L_j] = 0$  for all  $i \neq j$ , so  $[L, L] = \bigoplus_{i=1}^n \bigoplus_{j=1}^n [L_i, L_j] = \bigoplus_{i=1}^n L_i = L$ . □

### 3 Representations of Lie algebras: basic terminology

For this section, we fix an arbitrary field  $\mathbb{F}$  and define all vector spaces and Lie algebras relative to  $\mathbb{F}$ . Suppose  $L$  is a Lie algebra.

An *L-representation* is a Lie algebra morphism  $\phi : L \rightarrow \mathfrak{gl}(V)$  for some vector space  $V$ .

Explicitly, this means a linear map  $\phi$  with  $\phi([X, Y]) = [\phi(X), \phi(Y)]$  for all  $X, Y \in L$ .

An *L-module* is a vector space  $V$  with a bilinear operation

$$\begin{aligned} L \times V &\rightarrow L \\ (X, v) &\mapsto X \cdot v \end{aligned}$$

such that  $[X, Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v)$  for all  $X, Y \in L$  and  $v \in V$ .

$L$ -representations and  $L$ -modules are equivalent notions, just using different syntax.

Any  $L$ -representation can be converted to an  $L$ -module:

**Proposition.** If  $\phi : L \rightarrow \mathfrak{gl}(V)$  is an  $L$ -representation then  $V$  is an  $L$ -module for the action

$$X \cdot v \stackrel{\text{def}}{=} \phi(X)v \quad \text{for } X \in L \text{ and } v \in V.$$

*Proof.* The module action is bilinear since  $\phi$  is linear.

Let  $v \in V$ . Since for any  $A, B \in \mathfrak{gl}(V)$  we have  $A(Bv) = (AB)v$ , it follows for  $X, Y \in L$  that

$$X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = (\phi(X)\phi(Y) - \phi(Y)\phi(X))v = [\phi(X), \phi(Y)]v = \phi([X, Y])v = [X, Y] \cdot v$$

as needed. □

Likewise, any  $L$ -module can be converted to an  $L$ -representation:

**Proposition.** If  $V$  is an  $L$ -module then the map

$$\phi : L \rightarrow \mathfrak{gl}(V)$$

defined by  $\phi(X) : v \mapsto X \cdot v$  for  $X \in L$  is an  $L$ -representation.

The proof is similar to the previous proposition.

The operations going between  $L$ -representations and  $L$ -modules are inverses of each other.

A *submodule* of an  $L$ -module  $V$  is a subspace  $U \subseteq V$  such that  $X \cdot u \in U$  for all  $X \in L$  and  $u \in U$ .

Any submodule is itself an  $L$ -module.

A *morphism*  $f : V \rightarrow W$  of two  $L$ -modules is a linear map such that

$$f(X \cdot v) = X \cdot f(v) \quad \text{for all } v \in V \text{ and } X \in L.$$

The *kernel* of such a morphism is the subspace

$$\ker(f) = \{v \in V : f(v) = 0\}.$$

This is a submodule since if  $f(v) = 0$  then  $f(X \cdot v) = X \cdot f(v) = X \cdot 0 = 0$  (as the  $\cdot$  action is bilinear).

An  $L$ -module morphism  $V \rightarrow W$  is an *isomorphism* if it is a bijection.

An  $L$ -module  $V$  is *irreducible* if only submodules are  $0$  and  $V \neq 0$ .

If  $V$  and  $W$  are  $L$ -modules then the vector space direct sum  $V \oplus W$  is also an  $L$ -module for the action

$$X \cdot (v + w) = X \cdot v + X \cdot w.$$

An  $L$ -module  $V$  is *completely reducible* if there are irreducible  $L$ -submodules  $V_i \subseteq V$  such that  $V = \bigoplus_i V_i$ .

The  $L$ -module  $0$  is not considered irreducible because we want the direct sum decomposition of a completely reducible  $L$ -module to be unique (up to isomorphism and rearrangement of irreducible factors).

A *scalar map*  $V \rightarrow V$  is a linear map of the form  $v \mapsto \lambda v$  for some fixed  $\lambda \in \mathbb{F}$ .

An  $L$ -representation is *irreducible* if the associated  $L$ -module structure on  $V$  is an irreducible module.

We state the following fundamental result without proof.

**Theorem** (Schur's lemma). Assume  $\mathbb{F}$  is algebraically closed with  $\text{char}(\mathbb{F}) = 0$ .

Suppose  $\phi : L \rightarrow \mathfrak{gl}(V)$  is an irreducible  $L$ -representation.

Then the only linear maps  $f : V \rightarrow V$  with  $f\phi(X) = \phi(X)f$  for all  $X \in L$  are the scalar maps  $V \rightarrow V$ .

Continue to suppose  $V$  is an  $L$ -module. Define  $V^*$  to be the vector space of all linear maps  $V \rightarrow \mathbb{F}$ .

**Proposition.** The vector space  $V^*$  is an  $L$ -module for the action

$$X \cdot f = \left( \text{the linear map } V \rightarrow \mathbb{F} \text{ sending } \boxed{v \mapsto -f(X \cdot v)} \right) \quad \text{for } X \in L \text{ and } f \in V^*.$$

Note the unexpected minus sign in this formula.

*Proof.* Checking that the  $L$ -action on  $V^*$  is bilinear is straightforward.

Let  $X, Y \in L$  and  $f \in V^*$ . Then for any  $v \in V$  we have

$$(X \cdot (Y \cdot f))(v) = -(Y \cdot f)(X \cdot v) = f(Y \cdot (X \cdot v)) \quad \text{and similarly} \quad (Y \cdot (X \cdot f))(v) = f(X \cdot (Y \cdot v)).$$

On the other hand

$$([X, Y] \cdot f)(v) = -f([X, Y] \cdot v) = -f(X \cdot (Y \cdot v) - Y \cdot (X \cdot v)) = f(Y \cdot (X \cdot v)) - f(X \cdot (Y \cdot v)).$$

Comparing formulas shows that  $[X, Y] \cdot f = Y \cdot (X \cdot f) - X \cdot (Y \cdot f)$  as needed. □

Suppose  $V$  and  $W$  are two  $L$ -modules, say with bases  $\{v_i\}_{i \in I}$  and  $\{w_j\}_{j \in J}$ .

The *tensor product*  $V \otimes W$  is the vector space spanned by all *tensors*  $v \otimes w$  with  $v \in V$  and  $w \in W$  where

$$(v + v') \otimes w = v \otimes w + v' \otimes w \quad \text{and} \quad v \otimes (w + w') = v \otimes w + v \otimes w' \quad \text{and} \quad (av) \otimes w = v \otimes (aw)$$

for  $v, v' \in V$  and  $w, w' \in W$  and  $a \in \mathbb{F}$ .

Nontrivial fact: the vector space  $V \otimes W$  has a basis given by  $\{v_i \otimes w_j\}_{(i,j) \in I \times J}$ .

**Proposition.** The tensor product  $V \otimes W$  is an  $L$ -module for the bilinear action satisfying

$$X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot w) \quad \text{for } X \in L, v \in V, \text{ and } w \in W.$$

*Proof.* If  $X, Y \in L$ ,  $v \in V$ , and  $w \in W$  then we can expand

$$[X, Y] \cdot (v \otimes w) = X \cdot (Y \cdot v) \otimes w - Y \cdot (X \cdot v) \otimes w + v \otimes X \cdot Y(\cdot w) - v \otimes Y \cdot (X \cdot w)$$

while

$$X \cdot (Y \cdot (v \otimes w)) - Y \cdot (X \cdot (v \otimes w)) = X \cdot (Y \cdot v \otimes w + v \otimes Y \cdot w) - Y \cdot (X \cdot v \otimes w + v \otimes X \cdot w).$$

If we further expand the second expression and cancel some terms then we recover the first expression.  $\square$

We mention one final property today.

**Proposition.** Let  $V$  be a finite-dimensional vector space. Then the linear map  $\Phi : V^* \otimes V \rightarrow \mathfrak{gl}(V)$  with

$$\Phi(f \otimes v) = \left( \text{the linear map } V \rightarrow V \text{ sending } \boxed{x \mapsto f(x)v} \right) \quad \text{for } f \in V^* \text{ and } v \in V$$

is an isomorphism of vector spaces.

*Proof.* As  $\dim(\mathfrak{gl}(V)) = \dim(V)^2 = \dim(V^*) \dim(V) = \dim(V^* \otimes V)$  we just need to show  $\Phi$  is surjective.

We may assume  $V = \mathbb{F}^n$  is the space of  $n$ -row column vectors.

Then we can think of  $V^*$  as the space of  $1 \times n$  matrices and  $\mathfrak{gl}(V)$  as the space of all  $n \times n$  matrices.

In this setup  $\Phi$  is just the out-of-order matrix multiplication map

$$\Phi : \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \mapsto \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \in \mathfrak{gl}(V).$$

By choosing  $a_i, b_j \in \{0, 1\}$  appropriately the right side can give any elementary matrix  $E_{ij} \in \mathfrak{gl}(V)$ .

Hence by linearity every  $n \times n$  matrix occurs in the image of  $\Phi$ , so  $\Phi$  is surjective.  $\square$

If  $V$  is a finite-dimensional  $L$ -module, then there is a unique  $L$ -module structure on  $\mathfrak{gl}(V)$  that makes  $\Phi$  into a module isomorphism. Working through the definitions shows that this module has the formula

$$X \cdot f = \left( \text{the linear map } V \rightarrow V \text{ sending } \boxed{v \mapsto X \cdot f(v) - f(X \cdot v)} \right) \quad \text{for } X \in L \text{ and } f \in \mathfrak{gl}(V).$$