1 Review: representations and Schur's lemma

Throughout, let L be a semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$.

Assume $\dim(L) < \infty$.

Last time we covered some basic concepts in the representation theory of Lie algebras.

Goals for today: discuss the *Casimir element* as a tool for proving *Weyl's theorem*, which states that all finite-dimensional representations of L are *completely reducible*.

Assume V is an $\mathbb{F}$-vector space with $\dim(V) < \infty$.

Recall that *Schur's Lemma* says that if $\phi : L \to \mathfrak{gl}(V)$ is an irreducible L-representation (meaning that the corresponding L-module structure on V is irreducible), then the only linear maps $f : V \to V$ satisfying

$$f \circ \phi(x) = \phi(x) \circ f \quad \text{for all } x \in L$$

are the *scalar maps* $f_\lambda : V \to V$ sending $v \mapsto \lambda v$ for some fixed $\lambda \in \mathbb{F}$.

2 Casimir elements

Assume $\phi : L \to \mathfrak{gl}(V)$ is an L-representation. Define $\beta : L \times L \to \mathbb{F}$ by the formula

$$\beta(X, Y) = \mathsf{trace}(\phi(X)\phi(Y)).$$

This is a bilinear form that is both symmetric and *associative* in the sense that

$$\beta([X, Y], Z) = \beta(X, [Y, Z]) \quad \text{for all } X, Y, X \in L.$$

The Killing form of L is the special case of β when ϕ is the adjoint representation $\mathsf{ad} : L \to \mathfrak{gl}(L)$.

The *radical* of β is the ideal $\mathsf{Rad}(\beta) = \{X \in L : \beta(X, Y) = 0 \text{ for all } Y \in L\}$.

We abbreviate by setting $S = \mathsf{Rad}(\beta)$. The form β is *non-degenerate* if $S = 0$.

An L-representation $\phi : L \to \mathfrak{gl}(V)$ is *faithful* if $\ker(\phi) = 0$, or equivalently if ϕ is injective.

Lemma. Assume ϕ is faithful. Then S is a solvable ideal of L.

Proof. In this case $S \cong \phi(S)$ since ϕ is injective, and Cartan's Criterion holds for $\phi(S)$ as we have

$$\mathsf{trace}(\phi(X)\phi(Y)) = \beta(X, Y) = 0 \quad \text{for all } X \in [S, S] \text{ and } Y \in S,$$

and so $\mathsf{trace}(AB) = 0$ for all $A \in [\phi(S), \phi(S)]$ and $B \in \phi(S)$. □

Proposition. Assume ϕ is faithful. Then β is non-degenerate.

Proof. Since we assume that L is semisimple, L has no nonzero solvable ideals, so $S = 0$. □

Going forward, we assume ϕ is faithful.

Then $\beta : L \times L \to L$ is a symmetric, associative, and non-degenerate bilinear form.

The next lemma depends only on these properties of β.

Let $\{X_i\}_{i\in I}$ be a basis of L and let $\{Y_i\}_{i\in I}$ be the dual basis with $\beta(X_i, Y_j) = \delta_{ij} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Here I is some arbitrary (finite) indexing set.

The dual basis $\{Y_i\}_{i\in I}$ is uniquely determined by $\{X_i\}_{i\in I}$ since β is non-degenerate.

Now define the *Casimir element* of ϕ to be $\boxed{c_\phi = \sum_{i\in I} \phi(X_i)\phi(Y_i)} \in \mathfrak{gl}(V)$.

Later, we will see that this element does not depend on the choice of basis $\{X_i\}_{i\in I}$.

Remark. This definition does require ϕ to be faithful.

When L is semisimple but $\psi : L \to \mathfrak{gl}(V)$ is an unfaithful representation, the *Casimir element* c_ψ is defined to be the Casimir element of the associated (faithful) representation of the quotient $L/\ker(\psi) \to \mathfrak{gl}(V)$.

Recall that all quotients of semisimle Lie algebras are semisimle.

Lemma. Fix $Z \in L$ and write $a_{ij}, b_{ij} \in \mathbb{F}$ for the coefficients with

$$[Z, X_i] = \textstyle\sum_{j\in I} a_{ij} X_j \qquad \text{and} \qquad [Z, Y_i] = \sum_{j\in I} b_{ij} Y_j.$$

Then for all indices i and k it holds that $a_{ik} = -b_{ki}$.

Proof. Using the bilinearity and associativity of β we compute that

$$\begin{aligned} a_{ik} &= \textstyle\sum_j \beta(X_j, Y_k) a_{ij} \\ &= \beta([Z, X_i], Y_k) = -\beta([X_i, Z], Y_k) = -\beta(X_i, [Z, Y_k]) = -\textstyle\sum_j \beta(X_i, Y_j) b_{kj} = -b_{ki}. \end{aligned}$$

□

Proposition. We have $[\phi(Z), c_\phi] = 0$ for all $Z \in L$.

Thus the linear operator $c_\phi : V \to L$ commutes with the action of L on V determined by ϕ.

Proof. Using the preceding lemma, we compute that

$$\begin{aligned} [\phi(Z), c_\phi(\beta)] &= \sum_i [\phi(Z), \phi(X_i)\phi(Y_i)] \\ &= \sum_i [\phi(Z), \phi(X_i)]\phi(Y_i) + \sum_i \phi(X_i)[\phi(Z), \phi(Y_i)] = \sum_{i,j} (a_{ij} + b_{ij})\phi(X_i)\phi(Y_j) = 0. \end{aligned}$$

□

Corollary. The following properties hold for the Casimir element of ϕ:

(a) One always has $\mathsf{trace}(c_\phi) = \dim(L)$.

(b) If ϕ is irreducible then $c_\phi = \frac{\dim(L)}{\dim(V)} \cdot \mathrm{id}_V$ is a scalar map.

Proof. For part (a), notice that

$$\mathsf{trace}(c_\phi) = \textstyle\sum_i \mathsf{trace}(\phi(X_i)\phi(Y_i)) = \sum_i \underbrace{\beta(X_i, Y_i)}_{=1} = \dim(L).$$

If ϕ is irreducible then c_ϕ acts as a scalar by Schur's Lemma in view of the preceding proposition.

This scalar is $\frac{\dim(L)}{\dim(V)}$ by part (a) since $\mathrm{char}(\mathbb{F}) = 0$. $\square$

Part (b) of this corollary indicates that c_ϕ is independent of the basis $\{X_i\}$ at least when ϕ is irreducible.

Example. Let $L = \mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{F} \right\}$. Remember that we assume $\mathrm{char}(\mathbb{F}) = 0$.

This (simple, hence also semisimple) Lie algebra has basis

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Take $V = \mathbb{F}^2$ and let $\phi = \mathrm{id} : L \to \mathfrak{gl}(V)$.

The basis dual to E, H, F with respect to the bilinear form $\beta(X, Y) = \mathsf{trace}(XY)$ is F, $\frac{1}{2}H$, E. Therefore

$$c_\phi = EF + \tfrac{1}{2}H^2 + FE = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}.$$

This is a scalar matrix, as we expect since $\mathbb{F}^2$ is an irreducible $\mathfrak{sl}_2(\mathbb{F})$-module.

3 Weyl's Theorem

We mention one miscellaneous lemma before getting to today's main theorem.

Lemma. Let $\phi : L \to \mathfrak{gl}(V)$ be a finite-dimensional (not necessarily faithful) representation of a finite-dimensional semisimple Lie algebra L. Then $\phi(L) \subseteq \mathfrak{sl}(V)$. Hence if $\dim(V) = 1$ then $\phi(L) = 0$.

Proof. $L = [L, L]$ by semisimplicity so $\phi(L) = \phi([L, L]) = [\phi(L), \phi(L)] \subseteq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$. $\square$

The following result is called *Weyl's theorem*. Combining this with our observation that c_ϕ acts as a scalar in the irreducible case shows that all Casimir elements of a given finite-dimensional L-representation ϕ coincide when L is semisimple, that is, c_ϕ does not depend on the choice of basis $\{X_i\}_{i \in I}$ for L.

Theorem. Suppose $\phi : L \to \mathfrak{gl}(V)$ is an L-representation, with L semisimple and $\dim(V) < \infty$. Then ϕ is *completely reducible*, meaning there exist irreducible L-submodules $V_1, V_2, \dots, V_n$ such that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n.$$

Proof. By replacing L with $L/\ker(\phi)$, we may assume that ϕ is faithful.

Suppose $W \subseteq V$ is a proper L-submodule.

By induction on dimension we just need to find a complementary L-submodule U such that $V = W \oplus U$.

We can certainly find a complementary subspace U such that $V = W \oplus U$ as vector spaces.

The hard part is to find such a subspace that is an L-module.

The argument to construct U has three steps. Here is the first one:

Claim. Assume W is irreducible and $\dim(V) = \dim(W) + 1$ so that $V = W \oplus \mathbb{F}$ as vector spaces. Then there exists a complementary L-submodule U such that $V = W \oplus U$.

Proof of the first claim. Let $c = c_\phi$ be the Casimir element of ϕ.
Then $v \mapsto cv$ is an L-module endormorphism of V.
Since W is an L-submodule and c is in the subalgebra of $\mathfrak{gl}(V)$ generated by $\phi(L)$, we have $cW \subseteq W$.
Additionally the subspace $\ker(c) = \{v \in V : czv = 0\}$ is an L-submodule.

By the lemma all 1-dimensional representations of semisimple Lie algebras are trivial.
Therefore L acts trivially on $V/W = \mathbb{F}$.
This means $c(V/W) = 0$ so $cV \subseteq W$. Therefore $\dim(\ker(c)) \geq 1$.
But c acts on W as a scalar by Schur's lemma, and this scalar cannot be zero since

$$\mathsf{trace}(c \text{ as a map } W \to W) = \mathsf{trace}(c \text{ as a map } V \to V) = \dim(L) \neq 0.$$

Hence $\ker(c) \cap W = 0$.
As $\dim(\ker(c)) + \dim(W) \geq \dim(V)$ we must have $V = W \oplus \ker(c)$.
Thus the desired complementary L-submodule is given by $U = \ker(c)$. ∎

Here is the next step:

Claim. Assume that $\dim(V) = \dim(W)+1$ so that $V = W \oplus \mathbb{F}$ as vector spaces but W is not irreducible. Then there exists a complementary L-submodule U such that $V = W \oplus U$.

Proof of the second claim. In this case there is a nonzero proper submodule $W' \subset W$.
By induction on dimension $V/W' = W/W' \oplus \tilde{U}$ for some L-submodule $\tilde{U} \subset V/W'$.

Define V' to be the preimage of $\tilde{U}$ under the quotient map $V \to V/W'$.
Then $V' \subset V$ is a nonzero proper L-submodule containing W' and we have $\tilde{U} = V'/W'$.

By induction on dimension $V' = W' \oplus U$ for some L-module U.
Then $\tilde{U} = V'/W' \cong U$ so we have

$$\dim(V) - \dim(W') = \dim(V/W') = \dim(W/W') + \dim(\tilde{U}) = \dim(W) - \dim(W') + \dim(U)$$

and thus $\dim(V) = \dim(W) + \dim(U)$.

Since $V/W' = W/W' \oplus V'/W'$ we have $W \cap U \subseteq W \cap V' \subseteq W'$.
Thus $W \cap U = (W \cap U) \cap W' = W \cap (U \cap W') = W \cap 0 = 0$.
We conclude that $V = W \oplus U$. ∎

Finally, we have the last step:

Claim. Assume that W is an arbitrary L-submodule with $\dim(V) \geq \dim(W) + 1$.

Then there exists a complementary L-submodule U such that $V = W \oplus U$.

Proof of the third claim. Write $X \cdot v = \phi(X)v$ for $X \in L$ and $v \in V$.

Let $\mathrm{Hom}(V, W)$ be the vector space of linear maps $f : V \to W$.

This becomes an L-module if we define $X \cdot f$ for $X \in L$ to be the map sending $v \mapsto X \cdot f(v) - f(X \cdot v)$.

Define $A = \{f \in \mathrm{Hom}(V, W) : f|_W \text{ is a scalar map}\}$ and $B = \{f \in \mathrm{Hom}(V, W) : f|_W = 0\}$.

Then $A \supseteq B$ are L-submodules and you can check that $L \cdot A \subseteq B$.

Any linear projection $V \to W$ is in A but not B, so $\dim(A) > \dim(B)$.

On the other hand if $f, g \in A$ and $g \notin B$ then $f \in \lambda g + B$ for some scalar $\lambda \in \mathbb{F}$ so $\dim(A/B) \leq 1$.

We conclude that $\dim(A) = \dim(B) + 1$.

Now by our second claim, there exists a 1-dimensional L-submodule $C \subseteq A$ with $A = B \oplus C$.

Choose a nonzero element $h \in C$. After rescaling we may assume that $h|_W = \mathrm{id}_W$.

Then $C = \mathbb{F}\text{-span}\{h\}$ and $\ker(h) = \{v \in V : h(v) = 0\}$ is a subspace with

$$V = \mathrm{image}(h) \oplus \ker(h) = W \oplus \ker(h).$$

The subspace $\ker(h)$ is in fact an L-submodule, since if $v \in \ker(h)$ and $X \in L$ then $X \cdot h \in C$ so

$$h(X \cdot v) = X \cdot h(v) - (X \cdot h)(v) = X \cdot 0 - 0 = 0,$$

which shows that $X \cdot v \in \ker(h)$. Thus the claim holds with $U = \ker(h)$. ■

The third claim completes our proof of the theorem. □