1 Review: Weyl's theorem

Let L be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$.

An *L-representation* is a Lie algebra morphism $\phi : L \to \mathfrak{gl}(V)$ for some vector space V.

An *L-module* is a vector space V with a bilinear map $L \times V \to V$, written $(X, v) \mapsto X \cdot v$, that has

$$[X, Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v) \quad \text{for all } X, Y \in L \text{ and } v \in V.$$

Assume L is *semisimple*, meaning that L has no nonzero solvable ideals. Recall that this implies that

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

where each L_i is a simple ideal (that is, each L_i is non-abelian and contains no proper ideals).

Theorem (Weyl's Theorem)**.** Every finite-dimensional L-module is completely reducible.

Also proved last time: if $\phi : L \to \mathfrak{gl}(V)$ is any L-representation, then $\phi(L) \subseteq \mathfrak{sl}(V) \subseteq \mathfrak{gl}(V)$.

Consequently, if $\phi : L \to \mathfrak{gl}(V)$ is an L-representation and $\dim(V) = 1$, then $\phi(L) = 0$ since $\mathfrak{sl}(V) = 0$.

2 Abstract Jordan decomposition

Continue to assume L is finite-dimensional and semisimple.

As $Z(L) = \{X \in L : \mathsf{ad}_X = 0\}$ is a solvable ideal, we have $Z(L) = 0$ so $\mathsf{ad} : L \to \mathfrak{gl}(L)$ is faithful.

Recall that if V is a vector space with $\dim(V) < \infty$ then $X \in \mathfrak{gl}(V)$ has a unique *Jordan decomposition*

$$X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$$

where $X_{\mathrm{ss}} \in \mathfrak{gl}(V)$ is semisimple, $X_{\mathrm{nil}} \in \mathfrak{gl}(V)$ is nilpotent, and $[X_{\mathrm{ss}}, X_{\mathrm{nil}}] = 0$.

Define the *abstract Jordan decomposition* of $X \in L$ to be

$$X = X^{\mathrm{abs}}_{\mathrm{ss}} + X^{\mathrm{abs}}_{\mathrm{nil}}$$

where $X^{\mathrm{abs}}_{\mathrm{ss}}, X^{\mathrm{abs}}_{\mathrm{nil}} \in L$ are the unique elements such that

$$\mathsf{ad}(X^{\mathrm{abs}}_{\mathrm{ss}}) = (\mathsf{ad}_X)_{\mathrm{ss}} \quad \text{and} \quad \mathsf{ad}(X^{\mathrm{abs}}_{\mathrm{nil}}) = (\mathsf{ad}_X)_{\mathrm{nil}}.$$

Here $(\mathsf{ad}_X)_{\mathrm{ss}}$ and $(\mathsf{ad}_X)_{\mathrm{nil}}$ are the semisimple and nilpotent parts of ad_X in $\mathfrak{gl}(L)$.

This new notation turns out to be redundant:

Theorem. Suppose $L \subseteq \mathfrak{gl}(V)$ for some finite-dimensional vector space V. Let $X \in L$.

Then the components X_{ss} and X_{nil} of the Jordan decomposition of X (as an element of $\mathfrak{gl}(V)$) are in L.

Hence $X_{\mathrm{ss}} = X^{\mathrm{abs}}_{\mathrm{ss}}$ and $X_{\mathrm{nil}} = X^{\mathrm{abs}}_{\mathrm{nil}}$ are also the components of the abstract Jordan decomposition of X.

Proof sketch. Given the properties of the Jordan decomposition, we just need to prove that $X_{\mathrm{ss}}, X_{\mathrm{nil}} \in L$.

This is nontrivial since although we know that X_{ss} and X_{nil} are polynomials in X, this does not immediately imply that $X_{\mathrm{ss}}, X_{\mathrm{nil}} \in L$ since L is not necessarily an associative subalgebra of $\mathfrak{gl}(V)$.

The vector space V is an L-module since $L \subseteq \mathfrak{gl}(V)$.

For each L-submodule $W \subseteq V$ define $L_W = \{Y \in \mathfrak{gl}(V) : Y \cdot W \subseteq W \text{ and } \mathsf{trace}_W(Y) = 0\}$.

Since $L = [L, L]$ as L is semisimple, we have $L \subseteq L_W$. Define

$$L' = \bigcap_W L_W \cap N_{\mathfrak{gl}(V)}(L) \subseteq L$$

where $N_{\mathfrak{gl}(V)}(L) = \{Y \in \mathfrak{gl}(V) : [Y, L] \subseteq L\}$ and the intersection is over all L-submodules $W \subseteq V$.

The Jordan components X_{ss} and X_{nil} are polynomials in X without constant term.

We therefore have $X_{\mathrm{ss}}, X_{\mathrm{nil}} \in L_W$ all L-submodules $W \subseteq V$ since

$$\mathsf{trace}_W(X_{\mathrm{ss}}) = \mathsf{trace}_W(X) \qquad \text{and} \qquad \mathsf{trace}_W(X_{\mathrm{nil}}) = 0.$$

We showed in an earlier lecture that $\mathsf{ad}_{X_{\mathrm{ss}}} = (\mathsf{ad}_X)_{\mathrm{ss}}$ and $\mathsf{ad}_{X_{\mathrm{nil}}} = (\mathsf{ad}_X)_{\mathrm{nil}}$.

Thus $\mathsf{ad}_{X_{\mathrm{ss}}}$ and $\mathsf{ad}_{X_{\mathrm{nil}}}$ are also polynomials in ad_X without constant term.

As $\mathsf{ad}_X(L) \subseteq L$ we conclude that $X_{\mathrm{ss}}, X_{\mathrm{nil}} \in N_{\mathfrak{gl}(V)}(L)$.

Thus $X_{\mathrm{ss}}, X_{\mathrm{nil}} \in L'$ and it suffices to show that $L = L'$.

One can derive this as a consequence of Weyl's theorem $\rightsquigarrow$ see the textbook for the details. □

From this point on, we write $X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$ for the abstract Jordan decomposition of $X \in L$.

This notation is unambiguous (even when $L \subseteq \mathfrak{gl}(V)$) by the previous theorem.

Theorem. Suppose $\phi : L \to \mathfrak{gl}(V)$ is a representation of a semisimple Lie algebra.

Assume $\dim(L) < \infty$ and $\dim(V) < \infty$.

Then for any $X \in L$ with abstract Jordan decomposition $X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$, the expression

$$\phi(X) = \phi(X_{\mathrm{ss}}) + \phi(X_{\mathrm{nil}})$$

is the Jordan decomposition of $\phi(X) \in \mathfrak{gl}(V)$.

Proof sketch. When $\phi = \mathsf{ad}$ this claim is how we define the abstract Jordan decomposition.

The preceding theorem covers the base case when $\phi = \mathrm{id}$.

For general ϕ, note that if $Y \in L$ is an eigenvector for $\mathsf{ad}_{X_{\mathrm{ss}}}$ then $\phi(Y)$ is an eigenvector for $\mathsf{ad}_{\phi(X_{\mathrm{ss}})}$ as

$$\mathsf{ad}_{\phi(X_{\mathrm{ss}})}(\phi(Y)) = [\phi(X_{\mathrm{ss}}), \phi(Y)] = \phi([X_{\mathrm{ss}}, Y]) = \phi(\mathsf{ad}_{X_{\mathrm{ss}}}(Y)).$$

Therefore $\phi(L)$ has a basis of eigenvectors for $\mathsf{ad}_{\phi(X_{\mathrm{ss}})}$ since L does for $\mathsf{ad}_{X_{\mathrm{ss}}}$.

Thus $\mathsf{ad}_{\phi(X_{\mathrm{ss}})}$ is semisimple. One can check similarly that $\mathsf{ad}_{\phi(X_{\mathrm{nil}})}$ is nilpotent, and we have

$$[\phi(X_{\mathrm{ss}}), \phi(X_{\mathrm{nil}})] = \phi([X_{\mathrm{ss}}, X_{\mathrm{nil}}]) = \phi(0) = 0.$$

As $\phi(L)$ is semisimple, our base case implies that $\phi(X_{\mathrm{ss}}) = \phi(X)_{\mathrm{ss}}$ and $\phi(X_{\mathrm{nil}}) = \phi(X)_{\mathrm{nil}}$. □

3 Representations of $\mathfrak{sl}_2(\mathbb{F})$

Continue to let $\mathbb{F}$ be algebraically closed with $\mathrm{char}(\mathbb{F}) = 0$. Recall that

$$\mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{F} \right\}$$

has standard basis

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

for which we have Lie bracket relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad \text{and} \quad [E, F] = H.$$

Our second goal today is to classify the irreducible modules of this simple Lie algebra.

Let V be a finite-dimensional $\mathfrak{sl}_2(\mathbb{F})$-module. Since ad_H is semisimple, the theorems in the preceding section imply that H acts on V as a semisimple operator, so we can decompose

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda \quad \text{where} \quad V_\lambda = \{v \in V : Hv = \lambda v\}.$$

This property relies of $\mathbb{F}$ being algebraically closed, so that all eigenvalues for H are present.

We refer to the eigenvalues λ for H as *weights* and the nonzero subspaces V_λ as *weight spaces*.

Lemma. If $v \in V_\lambda$ then $Ev \in V_{\lambda+2}$ and $Fv \in V_{\lambda-2}$.

Proof. We have $HEv = [H, E]v + EHv = (2 + \lambda)Ev$ and $HFv = [H, F]v + FHv = (-2 + \lambda)Fv$. □

Assume our $\mathfrak{sl}_2(\mathbb{F})$-module V is irreducible with $0 < \dim(V) < \infty$.

Then there must exist at least one $\lambda \in \mathbb{F}$ with $V_\lambda \neq 0 = V_{\lambda+2}$.

For this λ we have $Ev = 0$ for all $v \in V_\lambda$.

We call the nonzero elements of this weight space V_λ the *maximal weight vectors* of V.

Lemma. Choose a maximal weight vector $v_0 \in V_\lambda$ and define $v_{-1} = 0$ and $v_i = \frac{1}{i!}F^i v_0$ for $i \geq 0$. Then:

(a) $Hv_i = (\lambda - 2i)v_i$.

(b) $Fv_i = v_{i+1}$.

(c) $Ev_i = i(\lambda - i + 1)v_{i-1}$.

Proof. Part (a) follows from the previous lemma.

Part (b) holds by definition.

Part (c) follows as an exercise from (a) and (b) by induction on i. □

The nonzero vectors v_i are linearly independent since they are H-eigenvectors with distinct eigenvalues.

Since $\dim(V) < \infty$, there exists a smallest m such that $v_m \neq 0$ and $v_{m+1} = 0$. Then we must have

$$V = \mathbb{F}\text{-span}\{v_0, v_1, v_2, \ldots, v_m\}$$

since V is irreducible and the space on the right is an $\mathfrak{sl}_2(\mathbb{F})$-submodule by the previous lemma.

In the basis $v_0, v_1, v_2, \ldots, v_m$ for V the elements H, E, and F respectively act as diagonal, strictly upper-triangular, and strictly lower-triangular matrices.

Moreover, we have $0 = E0 = Ev_{m+1} = (m+1)(\lambda - m)v_m$ by the lemma.

Corollary. We have $\lambda = m \in \mathbb{Z}_{\geq 0}$ and so the weight of any highest weight vector in an irreducible, finite-dimensional $\mathfrak{sl}_2(\mathbb{F})$-module V is a nonnegative integer.

We call this integer λ the *highest weight* of the $\mathfrak{sl}_2(\mathbb{F})$-module V.

Theorem. Let V be an irreducible $\mathfrak{sl}_2(\mathbb{F})$-module with $\dim(V) = m + 1 < \infty$. Then:

(a) $V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$ where $V_i = \{v \in V : Hv = iv\}$ has $\dim(V_i) = 1$ for all

$$i \in \{-m, -m+2, \ldots, m-2, m\}.$$

(b) V has a unique highest weight space, and this weight space has weight m.

(c) For each $m \geq 0$ there is a unique irreducible $\mathfrak{sl}_2(\mathbb{F})$-module of dimension $m+1$, up to isomorphism.

Proof. Parts (a) and (b) were derived in the discussion above.

To prove part (c), it suffices to check existence, namely, that the formulas for the action of E, F, and H in the preceding lemma define an $\mathfrak{sl}_2(\mathbb{F})$-module for all $\lambda \in \mathbb{Z}_{\geq 0}$. For this, one needs to verify that

$$[X, Y] \cdot v_i = X \cdot (Y \cdot v_i) - Y \cdot (X \cdot v_i)$$

for all $X, Y \in \{E, F, H\}$, and this is a straightforward calculation. □

If $\dim(V) = m + 1$ is odd then V looks like

$$V_{-m} \xrightarrow{F} V_{-m+2} \xrightarrow{F} \cdots \xrightarrow{F} V_{-2} \xrightarrow{F} V_0 \xrightarrow{F} V_2 \xrightarrow{F} \cdots \xrightarrow{F} V_{m-2} \xrightarrow{F} V_m$$

while if $\dim(V) = m + 1$ is even the V looks like

$$V_{-m} \xrightarrow{F} V_{-m+2} \xrightarrow{F} \cdots \xrightarrow{F} V_{-1} \xrightarrow{F} V_1 \xrightarrow{F} \cdots \xrightarrow{F} V_{m-2} \xrightarrow{F} V_m.$$

Here each arrow is an isomorphism of 1-dimensional subspaces afforded by the action of the generator F.

Thus exactly one of V_0 or V_1 is nonzero (and 1-dimensional) when V is irreducible.

Since $\mathfrak{sl}_2(\mathbb{F})$ is semisimple (as it is simple), these observations plus Weyl's theorem imply the following:

Corollary. Let V be a finite-dimensional $\mathfrak{sl}_2(\mathbb{F})$-module.

Then $H \in \mathfrak{sl}_2(\mathbb{F})$ acts on V as a semisimple operator whose eigenvalues are integers.

If λ is one of these eigenvalues, then so is $-\lambda$. Finally, if

$$V_\lambda = \{v \in V : Hv = \lambda v\}$$

then the number of summands in any irreducible decomposition of V is $\dim(V_0) + \dim(V_1)$.