1 Review

Let L be a Lie algebra over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$.

Assume L is **finite-dimensional** and **semisimple**. Then $\mathsf{ad} : L \to \mathfrak{gl}(L)$ is faithful since $Z(L) = 0$.

1.1 Abstract Jordan decompositions

Recall that if V is a vector space with $\dim(V) < \infty$ then $X \in \mathfrak{gl}(V)$ has a unique *Jordan decomposition*

$$X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$$

where $X_{\mathrm{ss}} \in \mathfrak{gl}(V)$ is semisimple (i.e., diagonalizable), $X_{\mathrm{nil}} \in \mathfrak{gl}(V)$ is nilpotent, and $[X_{\mathrm{ss}}, X_{\mathrm{nil}}] = 0$.

The *abstract Jordan decomposition* of $X \in L$ is defined in terms of the Jordan decomposition of ad_X as

$$X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$$

where $X_{\mathrm{ss}} \in L$ and $X_{\mathrm{nil}} \in L$ are the unique elements with $\mathsf{ad}(X_{\mathrm{ss}}) = (\mathsf{ad}_X)_{\mathrm{ss}}$ and $\mathsf{ad}(X_{\mathrm{nil}}) = (\mathsf{ad}_X)_{\mathrm{nil}}$.

It is not obvious that L actually contains elements X_{ss} and X_{nil} with these properties.

However this holds (and the elements are unique) when L is semisimple.

Our notation for these Jordan decompositions seems like it would be ambiguous if $L \subseteq \mathfrak{gl}(V)$. However:

Theorem. When L is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional vector space V, the ordinary and abstract Jordan decompositions are the same for each $X \in L$.

1.2 Representations of $\mathfrak{sl}_2(\mathbb{F})$

Recall $\mathrm{char}(\mathbb{F}) = 0$ so

$$\mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}\text{-span}\left\{ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is both simple and semisimple.

Theorem. Each irreducible $\mathfrak{sl}_2(\mathbb{F})$-module V with $\dim(V) = m + 1 < \infty$ decomposes as

$V = \bigoplus_{\lambda \in \Lambda} V_\lambda$ where $\Lambda = \{-m, -m+2, \dots, m-2, m\}$ and each $V_\lambda = \{v \in V : Hv = \lambda v\}$ is 1-dim.

In this *weight space decomposition* E and F give (non-inverse) isomorphisms $V_\lambda \xrightarrow{E} V_{\lambda+2}$ and $V_\lambda \xrightarrow{F} V_{\lambda-2}$.

More precisely, there exists a basis $\{v_\lambda : \lambda \in \Lambda\}$ for V such that

$$Hv_\lambda = \lambda v_\lambda \quad \text{and} \quad Fv_\lambda = v_{\lambda-2} \quad \text{and} \quad Ev_\lambda \in \mathbb{Z}\text{-span}\{v_{\lambda+2}\} \setminus \{0\} \qquad \text{where } v_{m+2} = v_{-m-2} = 0.$$

Finally, for each $m \geq 0$ there is a unique irreducible $\mathfrak{sl}_2(\mathbb{F})$-module of dimension $m+1$ up to isomorphism.

Example. Since $\mathfrak{sl}_2(\mathbb{F})$ is simple, its adjoint representation on $V = \mathfrak{sl}_2(\mathbb{F})$ is irreducible with $\dim(V) = 3$.

Note that $\mathfrak{sl}_2(\mathbb{F}) = V_{-2} \oplus V_0 \oplus V_2$ where $V_{-2} = \mathbb{F}\text{-span}\{F\}$, $V_0 = \mathbb{F}\text{-span}\{H\}$, and $V_2 = \mathbb{F}\text{-span}\{E\}$ satisfy

$$V_\lambda = \{v \in \mathfrak{sl}_2(\mathbb{F}) : [H, v] = \lambda v\}.$$

2 Root space decompositions

Continue to assume L is a finite-dimensional, semisimple Lie algebra defined over $\mathbb{F}$.

As usual, $\mathbb{F}$ is an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$ unless otherwise noted.

Today we will construct the *root space decomposition* of L.

This will generalize the weight space decomposition $\mathfrak{sl}_2(\mathbb{F}) = V_{-2} \oplus V_0 \oplus V_2$ seen in the previous example.

An element $X \in L$ with abstract Jordan decomposition $X = X_{\mathrm{ss}} + X_{\mathrm{nil}}$ is *semisimple* if $X = X_{\mathrm{ss}}$.

This happens precisely when L has a basis of eigenvectors for ad_X.

A Lie subalgebra $T \subseteq L$ is *toral* if every element of T is semisimple.

Lemma. Any toral subalgebra $T \subseteq L$ is abelian, meaning $[X, Y] = 0$ for all $X, Y \in T$.

Proof. Assume $T \subseteq L$ is a toral subalgebra and let $X \in T$.

Then $\mathsf{ad}_X : L \to L$ is diagonalizable and preserves the subspace T.

Linear algebra exercise: therefore T has a basis of eigenvectors for ad_X.

Suppose $Y \in T$ and $[X, Y] = aY$ for some $a \in \mathbb{F}$.

Given the exercise, to prove that T is abelian, it is enough to show that $[X, Y] = 0$.

Notice that the exercise implies that X is a linear combination of eigenvectors for ad_Y.

If this linear combination does not involve all 0-eigenvectors for ad_Y, then

$$[Y, [Y, X]] = \mathsf{ad}_Y(\mathsf{ad}_Y(X)) \neq 0.$$

However, this is impossible since $[Y, [Y, X]] = [Y, -aY] = -a[Y, Y] = 0$.

Therefore $\mathsf{ad}_Y(X) = 0$ or equivalently $[X, Y] = -[Y, X] = 0$ as needed. □

Choose a *maximal* toral subalgebra $H \subseteq L$.

This means H is a toral subalgebra not properly contained in any other.

There may be multiple choices for H.

Example. If $L = \mathfrak{sl}_n(\mathbb{F})$ then one choice for H is the subalgebra of traceless diagonal matrices.

Since L is semisimple, the center $Z(L) = 0$ is trivial, so $H \neq L$.

Recall that an abelian Lie algebra is solvable.

Since L has no nonzero solvable ideals, a nonzero toral subalgebra is never an ideal.

Since the maximal toral subalgebra H is abelian, the family

$$\mathsf{ad}(H) = \{\mathsf{ad}_H : X \in H\}$$

consists of commuting semisimple linear maps $L \to L$.

Therefore L can be simultaneously diagonalized with respect to $\mathsf{ad}(H)$, meaning there is a decomposition

$$L = \bigoplus_{\alpha \in H^*} L_\alpha$$

where H^* is the vector space of linear maps $H \to \mathbb{F}$ and for each $\alpha \in H^*$ we set

$$L_\alpha = \{X \in L : [h, X] = \alpha(h)X \text{ for all } h \in H\}.$$

Notice that $L_0 = \{X \in L : [h, X] = 0 \text{ for all } h \in H\} = C_L(H)$ is the *centralizer* of H in L.

The subspace L_α may be zero. If $L_\alpha \neq 0$ and $\alpha \neq 0$ then α is called a *root*.

Let $\Phi = \{\alpha \in H^* \setminus \{0\} : L_\alpha \neq 0\}$ be the set of all roots for L relative to H.

This is a finite set since $\dim(L) < \infty$.

Definition. The decomposition

$$L = \bigoplus_{\alpha \in \{0\} \sqcup \Phi} L_\alpha = C_L(H) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

is called the *Cartan decomposition* or *root space decomposition* of L.

Example. Suppose $L = \mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}\text{-span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ and $H = \mathbb{F}\text{-span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$.

Then H^* is 1-dimensional and spanned by the linear map $\alpha : H \to \mathbb{F}$ with

$$\alpha(X) = X_{11} - X_{22} \qquad \text{which means that} \qquad \alpha\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 2.$$

We deduce that $\Phi = \{-\alpha, \alpha\}$ after checking that

$$C_L(H) = H \quad \text{and} \quad L_\alpha = \mathbb{F}\text{-span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} \quad \text{and} \quad L_{-\alpha} = \mathbb{F}\text{-span}\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

3 Properties of root spaces

We continue the setup of the previous section and develop some properties of the root spaces of L.

Proposition. For all $\alpha, \beta \in H^*$ it holds that $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

Proof. The Jacobi identity implies for any $h, X, Y \in L$ that

$$[h, [X, Y]] = -[[X, Y], h] = -[X, [Y, h]] + [Y, [X, h]] = [X, [h, Y]] - [Y, [h, X]].$$

When $h \in H$, $X \in L_\alpha$, and $Y \in L_\beta$ the last expression is equal to

$$\beta(h)[X, Y] - \alpha(h)[Y, X] = (\alpha(h) + \beta(h))[X, Y]$$

which shows that $[X, Y] \in L_{\alpha+\beta}$. □

Proposition. Suppose $X \in L_\alpha$ for some $0 \neq \alpha \in H^*$. Then ad_X is nilpotent.

Proof. If $Y \in L_\beta$ for some $\beta \in H^*$ then $(\mathsf{ad}_X)^n(Y) \in L_{n\alpha+\beta}$ by the previous result.

Since there are only finitely many roots, we have $L_{n\alpha+\beta} = 0$ for all $n \gg 0$.

As $L = \bigoplus_{\alpha\in\{0\}\sqcup\Phi} L_\alpha$ it follows that $(\mathsf{ad}_X)^n = 0$ for some sufficiently large integer $n > 0$. □

Recall the definition $\mathcal{K}(X,Y) = \mathrm{tr}(\mathsf{ad}_X \mathsf{ad}_Y)$ of the *Killing form* $L \times L \to \mathbb{F}$.

This bilinear form is nondegenerate—meaning $\mathcal{K}(X,\cdot)$ is nonzero when $X \neq 0$—since L is semisimple.

Proposition. Suppose $\alpha, \beta \in H^*$ and $\alpha + \beta \neq 0$. Then $\mathcal{K}(X,Y) = 0$ for all $X \in L_\alpha$ and $Y \in L_\beta$.

Thus the subspaces L_α and L_β are orthogonal with respect to $\mathcal{K}$.

Proof. First choose $h \in H$ with $(\alpha+\beta)(h) \neq 0$. This is possible since $\alpha + \beta \neq 0$.

Next suppose $X \in L_\alpha$ and $Y \in L_\beta$. Then the bilinearity and associativity of $\mathcal{K}$ imply that

$$\alpha(h)\mathcal{K}(X,Y) = \mathcal{K}([h,X],Y) = -\mathcal{K}([X,h],Y) = -\mathcal{K}(X,[h,Y]) = -\beta(h)\mathcal{K}(X,Y).$$

Thus $(\alpha+\beta)(h)\mathcal{K}(X,Y) = 0$ so $\mathcal{K}(X,Y) = 0$. □

Corollary. The Killing form $\mathcal{K}$ restricts to a nondegenerate bilinear form on

$$L_0 = C_L(H) = \{X \in L : [h,X] = 0 \text{ for all } h \in H\}.$$

Proof. Suppose $0 \neq X \in L_0$. Then $\mathcal{K}(X,Y) = 0$ for all $Y \in \bigoplus_{\alpha\in\Phi} L_\alpha$ by the previous result.

As $\mathcal{K}$ is nondegenerate on L, there must exist $Y \in L_0$ with $\mathcal{K}(X,Y) \neq 0$. □

Lemma. Let V be a finite-dimensional vector space. Suppose $X, Y \in \mathfrak{gl}(V)$ are such that $XY = YX$.

If Y is nilpotent, then XY is nilpotent and $\mathsf{trace}(XY) = \mathsf{trace}(Y) = 0$.

Proof. The operator XY is nilpotent since $(XY)^n = X^nY^n$.

Then recall the trace of any nilpotent linear operator is the sum of its eigenvalues, which are all 0. □

The last main result we will prove today is the following.

Theorem. Let H be a maximal toral subalgebra of a finite-dimensional semisimple Lie algebra L. Then

$$H = L_0 = C_L(H) = \{X \in L : [X,h] = 0 \text{ for all } h \in H\}.$$

Thus the Cartan decomposition of L simplifies to $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$.

Proof. Let $C = C_L(H)$. We prove that $C = H$ via a series of claims.

Claim. If $X \in C$ then its Jordan components X_{ss} and X_{nil} are both elements of C.

Proof of this claim. Let $X \in C$. This holds precisely when $X \in L$ and ad_X maps $H \to 0$.
Since $(\mathsf{ad}_X)_{\text{ss}}$ and $(\mathsf{ad}_X)_{\text{nil}}$ are polynomials in ad_X without constant term, they also map $H \to 0$.

But we know that $X_{\text{ss}} \in L$ and $X_{\text{nil}} \in L$ and that $(\mathsf{ad}_X)_{\text{ss}} = \mathsf{ad}_{X_{\text{ss}}}$ and $(\mathsf{ad}_X)_{\text{nil}} = \mathsf{ad}_{X_{\text{nil}}}$.
Thus we conclude that $X_{\text{ss}} \in C$ and $X_{\text{nil}} \in C$. ■

Claim. If $X \in C$ is semisimple then $X \in H$.

Proof of this claim. In this case $H + \mathbb{F}\text{-span}\{X\}$ is a toral subalgebra containing H.
This must be equal to H by maximality, so $X \in H$. ■

We have already shown that the restricted Killing form $\mathcal{K}|_{C\times C}$ is nondegenerate.
Since we don't yet know that $C = H$, the following claim needs proof:

Claim. $\mathcal{K}|_{H\times H}$ is nondegenerate.

Proof of this claim. Suppose $h \in H$ and $\mathcal{K}(h, H) = 0$. We just need to show that $h = 0$.
Consider $X \in C$. By the first two claims we have $X_{\text{nil}} \in C$ and $X_{\text{ss}} \in H \subseteq C$.
Therefore $\mathcal{K}(h, X_{\text{ss}}) = 0$ so $\mathcal{K}(h, X) = \mathcal{K}(h, X_{\text{ss}}) + \mathcal{K}(h, X_{\text{nil}}) = \mathsf{trace}(\mathsf{ad}_h \mathsf{ad}_{X_{\text{nil}}})$.

The operators ad_h and $\mathsf{ad}_{X_{\text{nil}}}$ commute since $[\mathsf{ad}_h, \mathsf{ad}_{X_{\text{nil}}}] = \mathsf{ad}_{[h, X_{\text{nil}}]} = \mathsf{ad}_0 = 0$ as $X_{\text{nil}} \in C$.
Since $\mathsf{ad}_{X_{\text{nil}}} = (\mathsf{ad}_X)_{\text{nil}}$ is nilpotent we conclude that $\mathcal{K}(h, X) = \mathsf{trace}(\mathsf{ad}_h \mathsf{ad}_{X_{\text{nil}}}) = 0$.
Thus $\mathcal{K}(h, C) = 0$, which is only possible if $h = 0$ since $H \subseteq C$ and $\mathcal{K}|_{C\times C}$ is nondegenerate. ■

Claim. C is nilpotent, meaning ad_X restricts to a nilpotent linear map $C \to C$ for all $X \in C$.

Proof of this claim. If $X = X_{\text{ss}} \in C$ then $X \in H$ so ad_X restricts to the nilpotent map $C \to 0$.
If $X = X_{\text{nil}} \in C$ is nilpotent then $\mathsf{ad}_X = \mathsf{ad}_{X_{\text{nil}}} = (\mathsf{ad}_X)_{\text{nil}}$ is nilpotent.

Linear combinations of commuting nilpotent operators are nilpotent.
Hence ad_X restricts to a nilpotent linear map $C \to C$ for all $X \in C$. ■

Claim. $H \cap [C, C] = 0$.

Proof of this claim. The associativity of $\mathcal{K}$ implies that $\mathcal{K}(H, [C, C]) = \mathcal{K}([H, C], C) = \mathcal{K}(0, C) = 0$.
As $\mathcal{K}|_{H\times H}$ is nondegenerate, no nonzero element $X \in H$ can belong to $[C, C]$. ■

Claim. $[C, C] = 0$.

Proof of this claim. Assume $[C, C] \neq 0$.

Then C is nilpotent and contains $[C, C]$ as a nonzero ideal.

Hence, results proved when discussing Engel's theorem imply that some $0 \neq Z \in [C, C]$ has $[Z, C] = 0$.

This element $Z \in [C, C] \cap Z(C)$ cannot be semisimple as then $0 \neq Z = Z_{\text{ss}} \in H \cap [C, C] = 0$.

Thus $0 \neq Z_{\text{nil}} \in C$.

As $\mathsf{ad}_{Z_{\text{nil}}}$ is a polynomial in ad_Z without constant term, we then have $Z_{\text{nil}} \in Z(C)$.

This means that the nilpotent operator $\mathsf{ad}_{Z_{\text{nil}}} = (\mathsf{ad}_Z)_{\text{nil}}$ commutes with ad_X for all $X \in C$.

Thus $\mathcal{K}(Z_{\text{nil}}, C) = 0$ which contradicts the fact that $\mathcal{K}|_{C \times C}$ is nondegenerate.

This contradiction implies that $[C, C] = 0$. ■

Our final claim establishes the theorem:

Claim. $C = H$.

Proof of this claim. We know that $H \subseteq C$. Suppose $Z \in C$.

Then $Z_{\text{ss}} \in H$ and $Z_{\text{nil}} \in C$ by our first two claims.

The nilpotent operator $\mathsf{ad}_{Z_{\text{nil}}} = (\mathsf{ad}_Z)_{\text{nil}}$ commutes with ad_X for all $X \in C$ since $[C, C] = 0$.

Thus $\mathcal{K}(Z_{\text{nil}}, C) = 0$ so we must have $Z_{\text{nil}} = 0$ since $\mathcal{K}|_{C \times C}$ is nondegenerate.

Hence $Z = Z_{\text{ss}} \in H$ as needed. ■

□