1 Review: root space decomposition

Let L be a finite-dimensional, semisimple Lie algebra, over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$.

A subalgebra $T \subseteq L$ is *toral* if every element of T is semisimple.

Theorem. Let $T \subseteq L$ be any toral subalgebra and assume $H \subseteq L$ is a maximal toral subalgebra.

(a) T and H are abelian, meaning $[T, T] = [H, H] = 0$.

(b) H is self-centralizing, meaning $H = C_L(H) = \{X \in L : [X, h] = 0 \text{ for all } h \in H\}$.

(c) The Killing form $\mathcal{K}(X, Y) = \mathsf{trace}(\mathsf{ad}_X \mathsf{ad}_Y)$ on L restricts to a nondegenerate form on H.

Choose a maximal toral subalgebra $H \subseteq L$.

The corresponding *root space decomposition* of L is

$$\boxed{L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha}$$

where we define

$L_\alpha = \{X \in L : [h, X] = \alpha(h)X \text{ for all } h \in H\}$ for each $\alpha \in H^*$, and

Φ is the finite set $\alpha \in H^*$ with $\alpha \neq 0$ and $L_\alpha \neq 0$.

We call L_α a *root space* and $\alpha \in \Phi$ a *root*.

Example. Suppose $L = \mathfrak{sl}_3(\mathbb{F})$ is the Lie algebra of 3×3 traceless matrices.

For a maximal toral subalgebra is

$$H = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a + b + c = 0 \right\}.$$

Define linear maps $\varepsilon_i : H \to \mathbb{F}$ by $\varepsilon_i(h) = h_{ii}$ (the diagonal entry in row i).

Then $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in H^*$ but do not form a basis since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ as a map $H \to \mathbb{F}$.

However, we have

$$L = H \oplus \underbrace{\mathbb{F}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=L_{\varepsilon_1 - \varepsilon_2}} \oplus \underbrace{\mathbb{F}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=L_{\varepsilon_2 - \varepsilon_1}} \oplus \underbrace{\mathbb{F}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=L_{\varepsilon_1 - \varepsilon_3}} \oplus \underbrace{\mathbb{F}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=L_{\varepsilon_3 - \varepsilon_1}} \oplus \underbrace{\mathbb{F}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{=L_{\varepsilon_2 - \varepsilon_3}} \oplus \underbrace{\mathbb{F}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{=L_{\varepsilon_3 - \varepsilon_2}}.$$

Thus the set of roots for L corresponding to H is $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq 3,\ i \neq j\}$.

2 Orthogonality properties of roots

Continue to fix a finite-dimensional, semisimple Lie algebra $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$.

Here H is a maximal toral subalgebra of L and $\Phi \subset H^*$ is the corresponding set of roots.

Write $\mathcal{K}(X, Y) = \mathsf{trace}(\mathsf{ad}_X \mathsf{ad}_Y)$ for the Killing form on L.

As $\mathcal{K}|_{H \times H}$ is non-degenerate, when $\alpha \in H^*$ there is a unique $t_\alpha \in H$ with $\mathcal{K}(t_\alpha, \cdot) = \alpha(h)$ for all $h \in H$.

Proposition. It holds that $H^* = \mathbb{F}\text{-span}\{\alpha \in \Phi\} = \mathbb{F}\Phi$.

Proof. Otherwise, there would be some $0 \neq h \in H$ with $\alpha(h) = 0$ for all $\alpha \in \Phi$.

Then we would have $[h, L_\alpha] = 0$ for all $\alpha \in \Phi$ so $[h, L] = 0$ meaning $0 \neq h \in Z(L)$.

This is impossible since L is semisimple and therefore has no abelian ideals. □

Proposition. If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

Proof. By a result last time we know that $\mathcal{K}(L_\alpha, L_\beta) = 0$ if $\alpha, \beta \in \Phi$ have $\alpha + \beta \neq 0$ while $\mathcal{K}(L_\alpha, H) = 0$.

Hence if $\alpha \in \Phi$ but $-\alpha \notin \Phi$ then $\mathcal{K}(L_\alpha, L) = \mathcal{K}(L_\alpha, L_{-\alpha}) = 0$ contradicting the non-degeneracy of $\mathcal{K}$. □

Proposition. Let $\alpha \in \Phi$, $X \in L_\alpha$, and $Y \in L_{-\alpha}$. Then $[X, Y] = \mathcal{K}(X, Y)t_\alpha \in H$.

Proof. If $h \in H$ then

$$\mathcal{K}(h, [X, Y]) = \mathcal{K}([h, X], Y) = \alpha(h)\mathcal{K}(X, Y) = \mathcal{K}(t_\alpha, h)\mathcal{K}(X, Y) = \mathcal{K}(h, \mathcal{K}(X, Y)t_\alpha).$$

This implies that $\mathcal{K}(h, [X, Y] - \mathcal{K}(X, Y)t_\alpha) = 0$ for all $h \in H$.

Therefore $[X, Y] - \mathcal{K}(X, Y)t_\alpha = 0$ by the non-degeneracy of $\mathcal{K}$. □

Example. Again let $L = \mathfrak{sl}_3(\mathbb{F})$.

Then every root has the form $\alpha = \varepsilon_i - \varepsilon_j$ for $i \neq j$ and every root space $L_{\varepsilon_i - \varepsilon_j} = \mathbb{F}E_{ij}$ is 1-dimensional.

One can compute that $\mathcal{K}(E_{ij}, E_{ji}) = 4$.

Hence we have $t_{\varepsilon_i - \varepsilon_j} = \frac{1}{\mathcal{K}(E_{ij}, E_{ji})}[E_{ij}, E_{ji}] = \frac{1}{4}(E_{ii} - E_{jj})$. The same formula holds in $\mathfrak{sl}_n(\mathbb{F})$.

Proposition. If $\alpha \in \Phi$ then $[L_\alpha, L_{-\alpha}] = \mathbb{F}\text{-span}\{t_\alpha\} \neq 0$.

Proof. By the last proposition we just need to show that if $[L_\alpha, L_{-\alpha}] \neq 0$.

But if $0 \neq X \in L_\alpha$ and $\mathcal{K}(X, L_{-\alpha}) = 0$ then $\mathcal{K}(X, L) = 0$, which is impossible as $\mathcal{K}$ is non-degenerate. □

Proposition. If $\alpha \in \Phi$ then $\alpha(t_\alpha) = \mathcal{K}(t_\alpha, t_\alpha)$ is nonzero.

Proof. We can find $X \in L_\alpha$ and $Y \in L_{-\alpha}$ with $[X, Y] = t_\alpha$ by the previous proposition.

Suppose $\alpha(t_\alpha) = 0$. Then $[t_\alpha, X] = \alpha(t_\alpha)X = 0$ and $[t_\alpha, Y] = -\alpha(t_\alpha)Y = 0$

In this case ad_X, ad_Y, and ad_{t_α} generate a solvable subalgebra of $\mathfrak{gl}(L)$.

Thus by Lie's theorem there is a basis for L relative to which ad_X and ad_Y are have upper-triangular matrices, and then $\mathsf{ad}_{t_\alpha} = \mathsf{ad}_{[X,Y]} = [\mathsf{ad}_X, \mathsf{ad}_Y]$ has a strictly upper-triangular matrix.

This means that ad_{t_α} is nilpotent.

But ad_{t_α} is also semisimple since $t_\alpha \in H$

Hence we must have $\mathsf{ad}_{t_\alpha} = 0$ giving the contradiction $0 \neq t_\alpha \in Z(L) = 0$. □

Proposition. Let $\alpha \in \Phi$ and $H_\alpha = \frac{2}{\mathcal{K}(t_\alpha,t_\alpha)} t_\alpha$. Suppose $E_\alpha \in L_\alpha \setminus \{0\}$.

Then there is an element $F_\alpha \in L_{-\alpha}$ such that $H_\alpha = [E_\alpha, F_\alpha]$, and it holds that

$$\mathbb{F}\text{-span}\{E_\alpha, F_\alpha, H_\alpha\} \cong \mathfrak{sl}_2(\mathbb{F})$$

via the linear map sending $E_\alpha \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $F_\alpha \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $H_\alpha \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Proof. The existence of H_α and F_α follow from the previous two propositions.

Checking that the isomorphism $\mathbb{F}\text{-span}\{E_\alpha, F_\alpha, H_\alpha\} \cong \mathfrak{sl}_2(\mathbb{F})$ works out is a simple calculation. □

3 Integrality properties of roots

We continue the setup of the previous section, so $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$.

The next property takes a little more work to prove.

Theorem. Let $\alpha \in \Phi$. Then $\dim(L_\alpha) = 1$ and $\mathbb{F}\alpha \cap \Phi = \{-\alpha, \alpha\}$.

Proof. Choose $0 \neq E_\alpha \in L_\alpha$ and let F_α and $H_\alpha = [E_\alpha, F_\alpha]$ be as in the previous proposition.

Define $S_\alpha = \mathbb{F}\text{-span}\{E_\alpha, F_\alpha, H_\alpha = [E_\alpha, F_\alpha]\} \cong \mathfrak{sl}_2(\mathbb{F})$.

Then let $M = \bigoplus_{c\in\mathbb{F}\setminus\{0\}} L_{c\alpha} \oplus H = H \oplus L_\alpha \oplus L_{-\alpha} \oplus$ (possibly other root spaces).

The vector space M is an S_α-module with weights 0 and $2c$ for each $0 \neq c \in \mathbb{F}$ with $L_{c\alpha} \neq 0$ since

$$c\alpha(H_\alpha) = c\alpha\left(\tfrac{2}{\alpha(t_\alpha)} t_\alpha\right) = 2c.$$

Thus if $L_{c\alpha} \neq 0$ then we must have $c \in \frac{1}{2}\mathbb{Z}$ since all $\mathfrak{sl}_2$-weights are integers.

Every irreducible S_α-submodule of M of even highest weight contributes one dimension to the zero weight space of M. This weight space, which is just the 0-eigenspace of H_α, is exactly H.

But $S_\alpha \subseteq M$ is irreducible and $H = \ker(\alpha) \oplus \mathbb{F}H_\alpha$ and S_α acts as zero on $\ker(\alpha)$

Since $L_\alpha \subseteq S_\alpha$ and $L_{-\alpha} \subseteq S_\alpha$ it follows that $L_{c\alpha} = 0$ if c is an even integer with $c \notin \{-2, 0, 2\}$.

We conclude that $2\alpha \notin \Phi$. More generally, we conclude that if $\beta \in \Phi$ then $2\beta \notin \Phi$.

Since $\alpha \in \Phi$ we must therefore also have $\frac{1}{2}\alpha \notin \Phi$.

This means that if $L_{c\alpha} \neq 0$ then $c \neq \frac{1}{2}$ so $2c \neq 1$. Therefore 1 cannot occur as a weight for M.

Thus $M = H + S_\alpha = \ker(\alpha) \oplus \mathbb{F}H_\alpha \oplus \mathbb{F}E_\alpha \oplus \mathbb{F}F_\alpha$ so $L_\alpha = \mathbb{F}E_\alpha$ and $\dim(L_\alpha) = 1$. □

This shows that $\dim(L) = \dim(H) + |\Phi|$.

Recall that if $\alpha \in \Phi$ then $H_\alpha = \frac{2}{\mathcal{K}(t_\alpha,t_\alpha)} t_\alpha$ where $t_\alpha \in H$ satisfies $\mathcal{K}(t_\alpha, \cdot) = \alpha$.

Proposition. Suppose $\alpha, \beta \in \Phi$.

(a) Then $\beta(H_\alpha) \in \mathbb{Z}$ and $\beta - \beta(H_\alpha)\alpha \in \Phi$.

(b) If $\alpha + \beta \in \Phi$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

(c) If $\alpha + \beta \neq 0$ then there are integers $r, q \geq 0$ such that

$$(\beta + \mathbb{Z}\alpha) \cap \Phi = \{\beta + i\alpha : i \in \mathbb{Z} \text{ and } -r \leq i \leq q\} \quad \text{and} \quad \beta(H_\alpha) = r - q.$$

(d) L is generated by its root spaces L_α for $\alpha \in \Phi$ as a Lie algebra.

We call $\beta(H_\alpha) \in \mathbb{Z}$ a *Cartan integer*. We refer to the set $(\beta + \mathbb{Z}\alpha) \cap \Phi$ as the *α-root string through β*.

Proof. We will just show that part (c) holds as the other properties are easier.

Assume $\alpha + \beta \neq 0$ and set $K = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$.

Only finitely many terms in this sum are nonzero.

No multiple of α except $\pm\alpha$ is a root, so we have $\beta + i\alpha \neq 0$ for all $i \in \mathbb{Z}$.

The space K is a submodule of $S_\alpha \cong \mathfrak{sl}_2(\mathbb{F})$ and each subspace $L_{\beta+i\alpha}$ is either

zero if $\beta + i\alpha \notin \Phi$, or

1-dimensional if $\beta + i\alpha \in \Phi$, in which case $(\beta + i\alpha)(H_\alpha) = \beta(H_\alpha) + 2i$ since $\alpha(H_\alpha) = 2$.

In the second case $\beta(H_\alpha) + 2i$ is the weight of H_α on $L_{\beta+i\alpha}$.

All of these numbers have the same (even or odd) parity,

Hence exactly one of the numbers 0 or 1 can occur as a weight, so K is an irreducible S_α-module.

Thus if $r, q \in \mathbb{Z}_{\geq 0}$ are maximal with $\beta - r\alpha \in \Phi$ and $\beta + q\alpha \in \Phi$ then the corresponding weights

$$\beta(H_\alpha) - 2r \qquad \text{and} \qquad \beta(H_\alpha) + 2q$$

sum to zero, and (c) follows. □

Finally define $(\alpha, \beta) = \mathcal{K}(t_\alpha, t_\beta)$ for $\alpha, \beta \in H^*$.

Let $E_{\mathbb{Q}} = \mathbb{Q}\text{-span}\{\alpha \in \Phi\} \subset H^*$. This makes sense since $\mathbb{F}$ has characteristic zero so contains $\mathbb{Q}$.

Then let $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$. If $\mathbb{R} \subset \mathbb{F}$ then we can just set $E = \mathbb{R}\text{-span}\{\alpha \in \Phi\}$.

One can prove the following theorem using the results above (but we omit the details here).

The properties listed below will motivate the definition of a *root system* in the next lecture.

Theorem. The symmetric bilinear form $(\cdot, \cdot)$ restricts to a *positive definite* form on E, meaning that

$$(\alpha, \alpha) > 0 \qquad \text{for all } 0 \neq \alpha \in E.$$

In addition, the following properties hold:

(a) Φ spans E over $\mathbb{R}$.

(b) If $\alpha \in \Phi$ then $\mathbb{R}\alpha \cap \Phi = \{-\alpha, \alpha\}$.

(c) If $\alpha, \beta \in \Phi$ then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$.

(d) If $\alpha, \beta \in \Phi$ then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Example. Let $L = \mathfrak{sl}_n(\mathbb{F})$ and suppose $H \subseteq L$ is the subspace of diagonal matrices.

Then H is a maximal toral subalgebra and the corresponding set of roots for L is

$$\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq n,\ i \neq j\}$$

where ε_i is the map $X \mapsto X_{ii}$.

As noted earlier for $n = 3$, we have $t_{\varepsilon_i - \varepsilon_j} = \frac{1}{4}(E_{ii} - E_{jj})$. One can compute that

$$(\varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l) = \mathcal{K}(t_{\varepsilon_i - \varepsilon_j}, t_{\varepsilon_k - \varepsilon_l}) = \tfrac{1}{4}\langle \varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l \rangle \quad \text{where} \quad \langle \varepsilon_i, \varepsilon_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

with $\langle \cdot, \cdot \rangle$ bilinear. Thus $\frac{2(\varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l)}{(\varepsilon_k - \varepsilon_l, \varepsilon_k - \varepsilon_l)} = \langle \varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l \rangle \in \mathbb{Z}$ as predicted by the theorem.