1 Review: properties of roots

Let L be a finite-dimensional, semisimple Lie algebra, over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$.

Choose maximal toral subalgebra $H \subseteq L$,

This means every element of H acts as a semisimple operator in the adjoint representation of L.

For each $\alpha \in H^*$ define $L_\alpha = \{X \in L : [h, X] = \alpha(h)X \text{ for all } h \in H\}$.

Also let Φ be the finite set $\alpha \in H^*$ with $\alpha \neq 0$ and $L_\alpha \neq 0$.

Then we have a *root space decomposition* $\boxed{L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha}$.

Some properties of this decomposition proved last time:

- If $\alpha \in \Phi$ then $\dim(L_\alpha) = 1$ and $\mathbb{F}\alpha \cap \Phi = \{-\alpha, \alpha\}$. Hence $\dim(L) = \dim(H) + |\Phi|$.

  Also, if $\alpha, \beta \in \Phi$ then $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ with equality if $\alpha + \beta \in \Phi$.

- The Killing form $\mathcal{K}$ on L restricts to a nondegenerate form on H.

  Hence for each $\alpha \in \Phi$ there is a unique $t_\alpha \in H$ with $\alpha = \mathcal{K}(t_\alpha, \cdot)$.

  This element has $\mathcal{K}(t_\alpha, t_\alpha) \neq 0$ and it holds that $L_\alpha \oplus \mathbb{F}t_\alpha \oplus L_{-\alpha} \cong \mathfrak{sl}_2(\mathbb{F})$.

  The set Φ spans H^* and the elements t_α span H, though neither is a basis.

- Finally, define $(\alpha, \beta) = \mathcal{K}(t_\alpha, t_\beta)$ for $\alpha, \beta \in \Phi$. Then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$.

  Moreover, if $\alpha + \beta \neq 0$ then there are integers $r, q \geq 0$ such that

  $$(\beta + \mathbb{Z}\alpha) \cap \Phi = \{\beta + i\alpha : i \in \mathbb{Z} \text{ and } -r \leq i \leq q\} \quad \text{and} \quad \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)} = r - q.$$

These properties motivate the axiomatic definition of a *root system*, of which Φ is an example.

2 Abstract root systems

Let E be a finite-dimensional real vector space, so $E \cong \mathbb{R}^n$ for some n.

Assume E has a symmetric, positive definite bilinear form $(\cdot, \cdot)$, like the standard inner product on $\mathbb{R}^n$.

Being *symmetric* means $(\alpha, \beta) = (\beta, \alpha)$ and being *positive definite* means $(\alpha, \alpha) > 0$ if $0 \neq \alpha \in E$.

Recall that $\|\alpha\| = \sqrt{(\alpha, \alpha)}$ and $(\alpha, \beta) = \|\alpha\|\|\beta\| \cos(\theta)$ where θ is angle between α and β.

For $0 \neq \alpha \in E$ define $r_\alpha : E \to E$ by the formula

$$r_\alpha(\beta) = \Big(\text{the vector obtained by reflecting } \beta \text{ across the hyperplane } H_\alpha = \{v \in E : (\alpha, v) = 0\}\Big).$$

If $c \in \mathbb{R}$ is such that $\beta - c\alpha \in H_\alpha$ then $r_\alpha(\beta) = \beta - 2c\alpha$.

But $\beta - c\alpha \in H_\alpha \Rightarrow (\beta - c\alpha, \alpha) = 0 \Rightarrow (\beta, \alpha) = c(\alpha, \alpha) \Rightarrow c = \frac{(\beta,\alpha)}{(\alpha,\alpha)}$.

Thus the *reflection* $r_\alpha : E \to E$ belongs to $\mathrm{GL}(E)$ and has formula

$$r_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \quad \text{where we define} \quad \langle \beta, \alpha \rangle = \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)}.$$

Notice that

$$r_\alpha^{-1} = r_\alpha \qquad \text{and} \qquad r_{c\alpha} = r_\alpha \text{ if } 0 \neq c \in \mathbb{R} \qquad \text{and} \qquad (r_\alpha(\beta), r_\alpha(\gamma)) = (\beta, \gamma).$$

Also observe that $\boxed{\langle c\beta, \alpha \rangle = c\langle \beta, \alpha \rangle}$ but $\boxed{\langle \beta, c\alpha \rangle = \frac{1}{c}\langle \beta, \alpha \rangle}$ for scalars $0 \neq c \in \mathbb{R}$.

Definition. A subset $\Phi \subseteq E$ is a *root system* if:

(R1) $|\Phi| < \infty$ and $0 \notin \Phi$ and Φ spans E.

(R2) If $\alpha \in \Phi$ then $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$.

(R3) If $\alpha \in \Phi$ then $r_\alpha(\beta) \in \Phi$ for all $\beta \in \Phi$.

(R4) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

The *Weyl group* of Φ is $W = \langle r_\alpha : \alpha \in \Phi \rangle \subseteq \mathrm{GL}(E)$.

Since Φ is finite and spans E, and since each r_α defines a permutation of Φ, it follows that W is isomorphic to a subgroup of the symmetric group of all permutations of Φ. Thus the Weyl group has $|W| < \infty$.

If $\Phi' \subseteq E'$ is another root system, then an *isomorphism* $\Phi \to \Phi'$ is a linear bijection $f : E \to E'$ with

$$f(\Phi) = \Phi' \quad \text{and} \quad \langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle \text{ for all } \alpha, \beta \in \Phi.$$

An isomorphism of root systems does **not** have to be an isometry $E \xrightarrow{\sim} E'$ relative to the form $(\cdot, \cdot)$.

Intuitive idea for root system:

- Suppose W is any finite subgroup of $\mathrm{GL}(E)$ generated by reflections r_α.
- Consider the set of lines $\mathbb{R}\alpha$ for $\alpha \neq 0$ with $r_\alpha \in W$.
- Replace each of these lines by a pair of vectors α and $-\alpha$.
- Then, up to rescaling, the result is a root system with Weyl group W.

  Moreover, any root system arises in this way.

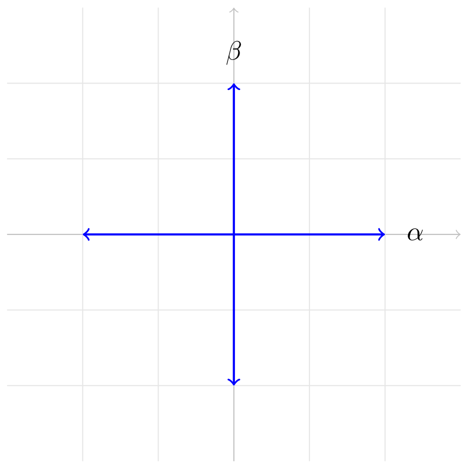
3 Examples of rank two root systems

Assume $E = \mathbb{R}^2$ with the standard inner product.

Here are four examples of root systems.

3.1 Type $A_1 \times A_1$

The root system $\Phi_{A_1 \times A_1} = \{\alpha, \beta, -\alpha, -\beta\}$ is shown below:



There are 4 roots in this system, and we have $(\alpha, \beta) = \langle \alpha, \beta \rangle = 0$.

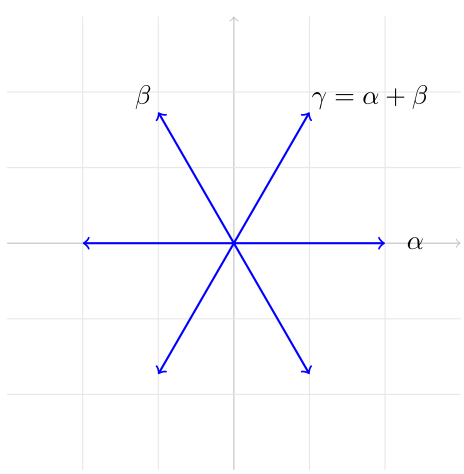
The reflection r_α negates $\pm\alpha$ and fixes $\pm\beta$. Similarly r_β negates $\pm\beta$ and fixes $\pm\alpha$.

Thus $r_\alpha r_\beta = r_\beta r_\alpha$ and $W = \langle r_\alpha, r_\beta \rangle \cong S_2 \times S_2$.

This example would also be a valid root system if we rescaled the length of α to be different than β.

3.2 Type A_2

The root system $\Phi_{A_2} = \{\alpha, \beta, \alpha + \beta, -\alpha - \beta, -\alpha - \beta\}$ is shown below:



There are 6 roots of equal length in this system.

Since $(\alpha, \beta) = \|\alpha\|\|\beta\| \cos \frac{2\pi}{3} = -\frac{\|\beta\|^2}{2}$ we have $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} = -1$.

One can similarly compute $\langle \alpha, \gamma \rangle \in \mathbb{Z}$ and $\langle \beta, \gamma \rangle \in \mathbb{Z}$ for $\gamma = \alpha + \beta$.

The reflection r_α negates $\pm\alpha$ and swaps $\pm\beta \leftrightarrow \pm\gamma$.

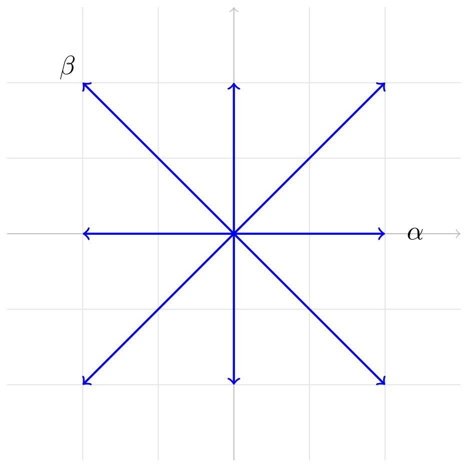
The reflection r_β negates $\pm\beta$ and swaps $\pm\alpha \leftrightarrow \pm\gamma$.

Finally, the reflection r_γ negates $\pm\gamma$ and swaps $\pm\alpha \leftrightarrow \mp\beta$.

One can check that $W = \langle r_\alpha, r_\beta, r_\gamma \rangle \cong S_3$.

3.3 Type B_2

The root system $\Phi_{B_2} = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, -\alpha, -\beta, -\alpha-\beta, -2\alpha-\beta\}$ is shown below:



There are 8 roots with two different lengths in this system.

We have $\|\beta\| = \sqrt{2}\|\alpha\| = \|2\alpha+\beta\|$ and $\|\alpha+\beta\| = \|\alpha\|$.

Thus $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} = \frac{2\|\alpha\|\|\beta\| \cos\frac{3\pi}{2}}{\|\beta\|^2} = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} = -1$ and likewise with other instances of $\langle \cdot, \cdot \rangle$.

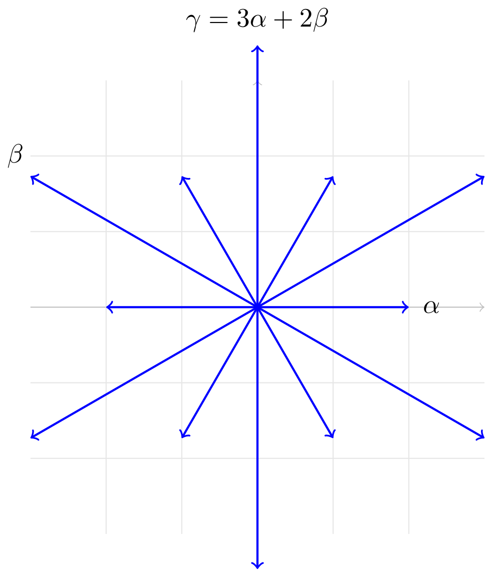
The reflection r_α negates $\pm\alpha$, swaps $\pm\beta \leftrightarrow \pm(2\alpha+\beta)$, and fixes $\pm(\alpha+\beta)$.

Similarly r_β negates $\pm\beta$, swaps $\pm\alpha \leftrightarrow \pm\pm(\alpha+\beta)$, and fixes $\pm(2\alpha+\beta)$.

One can show that $W = \langle r_\alpha, r_\beta \rangle \cong \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$ has $|W| = 8$.

3.4 Type G_2

Finally, we have the root system Φ_{G_2} is shown below:



There are 12 roots with two different lengths.

The 6 short roots look like Φ_{A_2} and the 6 long roots also look like Φ_{A_2}.

One can show that W is isomorphic to a dihedral group of order 12.

Notice that in all of these examples we have $|W| = |\Phi|$.

4 Pairs of roots

Suppose Φ is any root system in our real vector space E.

The *rank* of Φ is $\dim(E)$.

The examples above all have rank 2. (The only root system with rank 1 is $\{\pm\alpha\}$.)

Suppose $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$. Let θ be the angle between the two roots. Then

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta \in \mathbb{Z}, \qquad \text{and} \qquad \langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4(\cos\theta)^2 \in \mathbb{Z}.$$

As $(\cos\theta)^2 \in [0, 1]$, the only possibilities for $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$, θ, $\|\beta\|^2/\|\alpha\|^2$, and Φ are as follows:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$	Φ
0	0	$\pi/2$	(unconstrained)	$A_1 \times A_1$
1	1	$\pi/3$	1	A_2
-1	1	$2\pi/3$	1	A_2
1	2	$\pi/4$	2	B_2
-1	-2	$3\pi/4$	2	B_2
1	3	$\pi/6$	3	G_2
-1	-3	$5\pi/6$	3	G_2

So in fact the four examples given account for all possible rank two root systems (up to isomorphism):

Proposition. Any rank two root system Φ is isomorphic to exactly one of $\Phi_{A_1 \times A_1}$, Φ_{A_2}, Φ_{B_2}, or Φ_{G_2}.

5 Root strings

Let Φ be a root system in E with Weyl group W.

Assume $\sigma \in \mathrm{GL}(E)$ has $\sigma(\Phi) = \Phi$.

Proposition. Then $\sigma r_\alpha \sigma^{-1} = r_{\sigma(\alpha)}$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.

Proof. Compute $\sigma r_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma r_\alpha(\beta) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$.

Clearly $\sigma r_\alpha \sigma^{-1}$ preserves Φ and sends $\sigma(\alpha) \mapsto -\sigma(\alpha)$.

Also $\sigma r_\alpha \sigma^{-1}$ fixes the hyperplane $\sigma(H_\alpha)$ where $H_\alpha = \{v \in E : (v, \alpha) = 0\}$.

Thus, if we knew that $\sigma(H_\alpha) = H_{\sigma(\alpha)}$ then the desired formulas would following immediately.

We outsource this claim to the following lemma. □

Continue to assume $\sigma \in \mathrm{GL}(E)$ has $\sigma(\Phi) = \Phi$.

Lemma. If σ fixes a hyperplane $E \subseteq H$ while sending some $0 \neq \alpha \in E$ to $-\alpha$, then $H = H_\alpha$ and $\sigma = \sigma_\alpha$.

Proof idea. Define $\tau = \sigma\alpha$.

Then $\tau(\alpha) = \alpha$ and $\tau(\tau) = \tau$ and τ fixes H pointwise.

Choose a basis $v_1, v_2, \ldots, v_{n-1}$ for H and set $v_n = \alpha$.

Since $\alpha \notin H$, the vectors $v_1, v_2, \ldots, v_n$ form a basis for E.

But the matrix of τ in this basis is the identity matrix, so $\tau = 1$.

For a more detailed argument, see the textbook. □

Lemma. Let $\alpha, \beta \in \Phi$ be non-proportional, so that $\alpha \neq \pm\beta$.

(a) If $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Phi$.

(b) If $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Phi$.

Proof. Part (b) follows from part (a), swapping β and $-\beta$.

To prove part (a), note that $(\alpha, \beta) > 0$ implies $\langle \alpha, \beta \rangle > 0$.

The acute angle between α and β must be $\pi/3, \pi/4$, or $\pi/6$ (consider the 4 root systems in $\mathbb{R}^2$).

Since α and β are not orthogonal, we must have $\langle \alpha, \beta \rangle = 1$ or $\langle \beta, \alpha \rangle = 1$.

If $\langle \alpha, \beta \rangle = 1$ then $\alpha - \beta = \sigma_\beta(\alpha) \in \Phi$.

If $\langle \beta, \alpha \rangle = 1$ then $\alpha - \beta = -\sigma_\alpha(\beta) \in \Phi$. □

For $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$, the *α-string through β* is the set of roots $\{\beta + i\alpha : i \in \mathbb{Z} \cap \Phi\}$.

Proposition. There are integers $q, r \geq 0$ such that the α-string through β is exactly $\{\beta + i\alpha : -r \leq i \leq q\}$.

Proof. If there were any gaps in the string, then we could find $p, s \in \mathbb{Z}$ with $-r \leq p < s \leq q$ where

$$\beta + p\alpha \in \Phi \text{ and } \beta + s\alpha \in \Phi \qquad \text{but} \qquad \beta + (p+1)\alpha \notin \Phi \text{ and } \beta + (s-1)\alpha \notin \Phi.$$

In this case the previous lemma implies that $(\beta + p\alpha, \alpha) \geq 0 \geq (\beta + s\alpha, \alpha)$ which means that

$$((s-p)\alpha, \alpha) = |s-p|(\alpha, \alpha) \leq 0.$$

This is impossible as $(\alpha, \alpha) > 0$ since $(\cdot, \cdot)$ is positive definite. □

Corollary. The integers $r, q \geq 0$ such that the α-string through β is $\{\beta + i\alpha : -r \leq i \leq q\}$ satisfy

$$r - q = \langle \beta, \alpha \rangle \in \{0, \pm 1, \pm 2, \pm 3\}.$$

Thus every α-string has at most 4 elements.

Proof. The reflection r_α preserves the α-string through β since $r_\alpha(\beta + i\alpha) = \beta - (\langle \beta, \alpha \rangle + i)\alpha$.

Therefore we must have $r_\alpha(\beta + q\alpha) = \beta - r\alpha$.

But $r_\alpha(\beta + q\alpha) = \beta - \langle \beta, \alpha \rangle\alpha - q\alpha$, so $\langle \beta, \alpha \rangle = r - q$. □