1 Review: abstract root systems

Let E be a finite-dimensional real vector space with a symmetric, positive definite bilinear form $(\cdot,\cdot)$. (By appropriately choosing a basis, we could identity E with $\mathbb{R}^n$ with the standard inner product.)

For $0 \neq \alpha \in E$, let $r_\alpha : E \to E$ be the map $\boxed{r_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha}$ and set $\boxed{\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}}$.

Definition. A finite subset $\Phi \subseteq E \setminus \{0\}$ is a *root system* if:

(R1) E is spanned by Φ.

(R2) If $\alpha \in \Phi$ then $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$.

(R3) If $\alpha \in \Phi$ then $r_\alpha(\Phi) = \Phi$.

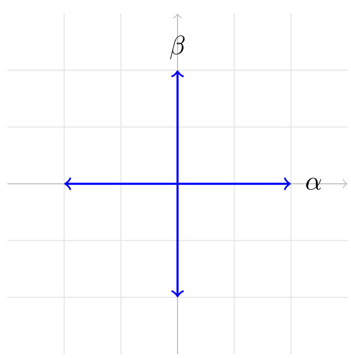
(R4) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

The elements of Φ are *roots*. The subgroup $W \subset \mathrm{GL}(E)$ generated by $\{r_\alpha : \alpha \in \Phi\}$ is the *Weyl group*.

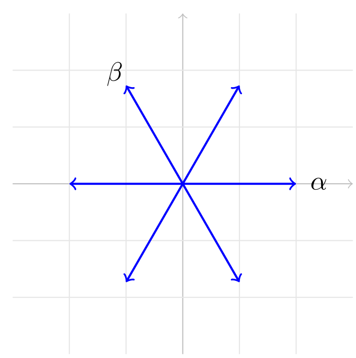
If $\Phi' \subseteq E'$ is another root system, then an *isomorphism* $\Phi \to \Phi'$ is a linear bijection $f : E \to E'$ with

$$f(\Phi) = \Phi' \quad \text{and} \quad \langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle \text{ for all } \alpha, \beta \in \Phi.$$

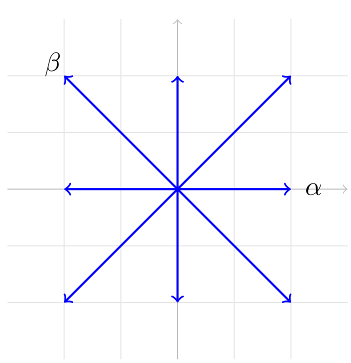
Up to isomorphism, there are 4 root systems in $\mathbb{R}^2$:



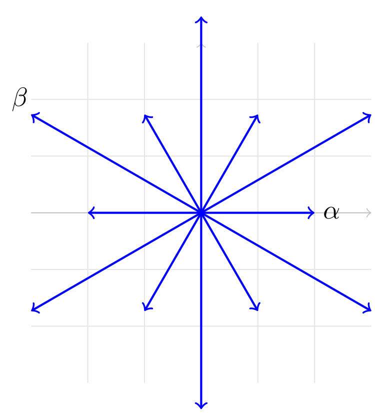
$\Phi_{A_1 \times A_1}$



Φ_{A_2}



Φ_{B_2}



Φ_{G_2}

2 Motivation

Suppose L is a (nonzero, finite-dimensional) semisimple Lie algebra over $\mathbb{C}$.

Choose a maximal toral subalgebra $H \subseteq L$ and let $H^* = \{\text{linear maps } H \to \mathbb{C}\}$.

(If L is classical, can take H to be the subalgebra of diagonal matrices in L.)

For each $\alpha \in H^*$ define as usual $L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$.

Set $\Phi = \{\alpha \in H^* \setminus \{0\} : L_\alpha \neq 0\}$.

We have seen that $H = L_0$ is abelian and that we have a decomposition $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$.

Here, $\boxed{\Phi \text{ is a root system}}$ in $E = \mathbb{R}\text{-span}\{\alpha \in \Phi\}$.

The relevant form $(\cdot, \cdot)$ is given by the restricted Killing form $\mathcal{K}|_{H\times H}$, transferred to H^* by non-degeneracy.

In other words, $(\alpha, \beta) = \mathcal{K}(t_\alpha, t_\beta)$ where $t_\alpha \in H$ is the unique element with $\mathcal{K}(t_\alpha, h) = \alpha(h)$ for all $h \in H$.

3 Simple roots and the Weyl group

Let Φ be a root system in a real vector space E with Weyl group W.

A *base* or *simple system* for Φ is an $\mathbb{R}$-basis Δ for E such that each $\alpha \in \Phi$ can be written as

$$\alpha = \textstyle\sum_{\beta\in\Delta} K_{\alpha\beta} \cdot \beta$$

where the coefficients $K_{\alpha\beta}$ are either

(i) all nonnegative integers or

(ii) all nonpositive integers.

Necessarily $|\Delta| = \dim(E)$. It takes some work to show that Φ always has a simple system.

However, such systems are never unique: if Δ is a simple system, then $-\Delta$ is another simple system.

Example. In each root system in $\mathbb{R}^2$, the roots labeled $\{\alpha, \beta\}$ form a simple system.

Lemma. Suppose Δ is a simple system for Φ and $\alpha, \beta \in \Delta$ have $\alpha \neq \beta$. Then $(\alpha, \beta) \leq 0$.

Proof. In this case we also have $\alpha \neq -\beta$ since the elements of Δ are linearly independent.

If $(\alpha, \beta) > 0$ then a lemma from last time says that $\alpha - \beta \in \Phi$.

But if $\alpha - \beta \in \Phi$ then Δ would not be a simple system. □

Fix a simple system Δ for Φ and define the *height* of a root $\alpha = \sum_{\beta\in\Delta} K_{\alpha\beta} \cdot \beta$ to be the sum

$$\boxed{\mathsf{ht}(\alpha) = \sum_{\beta\in\Delta} K_{\alpha\beta} \in \mathbb{Z} \setminus \{0\}}.$$

Let $\boxed{\Phi^+ = \{\alpha \in \Phi : \mathsf{ht}(\alpha) > 0\}}$ and $\boxed{\Phi^- = \{\alpha \in \Phi : \mathsf{ht}(\alpha) < 0\}}$. Then $\Phi^- = -\Phi^+$ and $\Phi = \Phi^+ \cup \Phi^-$.

Call Φ^+ the set of *positive roots* and Φ^- the set of *negative roots* corresponding to Δ.

Theorem. Any root system Φ does have a simple system.

To prove this theorem, we will instead derive a more detailed statement.
For each $\gamma \in E$ define $\Phi^+(\gamma) = \{\alpha \in \Phi : (\gamma, \alpha) > 0\}$ and $H_\gamma = \{v \in E : (\gamma, v) = 0\}$.
One can always choose $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ and we call such γ *regular*.

If γ is regular then $\Phi = \Phi^+(\gamma) \cup \Phi^-(\gamma)$ where $\Phi^-(\gamma) = -\Phi^+(\gamma)$.
Call $\alpha \in \Phi^+(\gamma)$ *indecomposable* if we cannot write $\alpha = \beta_1 + \beta_2$ where each $\beta_i \in \Phi^+(\gamma)$.

Theorem. If $\gamma \in E$ is regular, then the set $\Delta(\gamma)$ of indecomposable roots in Φ is a simple system.
Moreover, every simple system arises as $\Delta(\gamma)$ for some choice of regular $\gamma \in E$.

Proof. We derive the theorem from a series of claims.

Define $\Delta(\gamma)$ to be the roots in $\Phi^+(\gamma)$ that are indecomposable.

Claim. Each $\alpha \in \Phi^+(\gamma)$ is in $\mathbb{Z}_{\geq 0}$-span$\{\beta \in \Delta(\gamma)\}$.

Proof of this claim. Otherwise, choose $\alpha \in \Phi^+(\gamma)$ that is not in $\Delta(\gamma)$ with (α, γ) minimal.
Then $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \Phi^+(\gamma)$ so $(\alpha, \gamma) = (\beta_1, \gamma) + (\beta_2, \gamma)$.

As $(\beta_i, \gamma) > 0$ the minimality of (α, γ) implies that $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 0}$-span$\{\beta \in \Delta(\gamma)\}$.
But this means that $\alpha = \beta_1 + \beta_2$ is also in the $\mathbb{Z}_{\geq 0}$-span of $\Delta(\gamma)$, which is a contradiction. ∎

Claim. If $\alpha, \beta \in \Delta(\gamma)$ and $\alpha \neq \beta$ then $(\alpha, \beta) \leq 0$.

Proof of this claim. Otherwise $\alpha - \beta \in \Phi$ and $\beta \neq \pm\alpha$ so $\alpha - \beta$ or $\beta - \alpha$ is in $\Phi^+(\gamma)$.
But then $\alpha = \beta + (\alpha - \beta)$ or $\beta = \alpha + (\beta - \alpha)$ would be decomposable. ∎

Claim. The set $\Delta(\gamma)$ is linearly independent.

Proof of this claim. Suppose we can write

$$0 = \textstyle\sum_\alpha c_\alpha \alpha - \sum_\beta d_\beta \beta$$

where α, β range over disjoint subsets of $\Delta(\gamma)$ and $c_\alpha, d_\beta \geq 0$. Then

$$0 \leq (\textstyle\sum_\alpha c_\alpha \alpha, \sum_\alpha c_\alpha \alpha) = \left(\sum_\alpha c_\alpha \alpha, \sum_\beta d_\beta \beta\right) = \sum_{\alpha,\beta} \underbrace{c_\alpha d_\beta}_{\geq 0} \underbrace{(\alpha, \beta)}_{\leq 0} \leq 0$$

and so we must have all $c_\alpha = 0$. One similarly derives that all $d_\beta = 0$. ∎

Combining the preceding claims shows that the set $\Delta(\gamma)$ is a simple system for Φ. Finally:

Claim. Every simple system in Φ arises as $\Delta(\gamma)$ for some regular $\gamma \in E$.

Proof of this claim. Given a simple system Δ for Φ, we need to find γ with $\Delta = \Delta(\gamma)$.

Exercise: one can always find a regular γ with $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$.

Then $\Phi^{\pm} = \Phi^{\pm}(\gamma)$ so every $\alpha \in \Delta$ must be indecomposable with respect to γ.

This means $\Delta \subseteq \Delta(\gamma)$. As $|\Delta| = |\Delta(\gamma)| = \dim(E)$, must have $\Delta = \Delta(\gamma)$. ■

□

We refer to the elements of a simple system Δ as *simple roots*.

The hyperplanes $H_\alpha = \{v \in E : (v, \alpha) = 0\}$ for $\alpha \in \Phi$ divide E into finitely many regions.

We call the connected components of $E \setminus \bigcup_{\alpha\in\Phi} H_\alpha$ the *Weyl chambers* of E.

4 Properties of simple roots

Fix a simple system Δ of Φ and define Φ^+ and $\Phi^- = -\Phi^+$ relative to Δ.

We call the elements of Φ^+ *positive roots* and the elements of Φ^- *negative roots*.

Lemma. If $\alpha \in \Phi^+$ but $\alpha \notin \Delta$ then $\alpha - \beta \in \Phi^+$ for some $\beta \in \Delta$.

Proof. Suppose $(\alpha, \beta) \geq 0$ for all $\beta \in \Delta$.

Then the argument for the third claim in the previous proof shows that $\Delta \cup \{\alpha\}$ is linearly independent.

As this is impossible, must have $(\alpha, \beta) > 0$ for some $\beta \in \Delta$ and then $\alpha - \beta \in \Phi$.

Then $\mathsf{ht}(\alpha - \beta) = \mathsf{ht}(\alpha) - 1 \geq 0$.

This inequality must be strict since $\alpha \neq \beta$ as $\alpha \notin \Delta$, so $\alpha - \beta$ must be in Φ^+. □

The next corollary follows from the previous lemma by induction:

Corollary. Each $\alpha \in \Phi^+$ can be written $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ where $\alpha_i \in \Delta$ and where

$$\alpha_1 + \alpha_2 + \cdots + \alpha_j \in \Phi^+ \quad \text{for all } 1 \leq j \leq k.$$

Lemma. Let $\alpha \in \Delta$. Then $r_\alpha(\alpha) = -\alpha$ and $r_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$.

Proof. The formula $r_\alpha(\alpha) = -\alpha$ holds by the definition of r_α.

Fix $\beta \in \Phi^+ \setminus \{\alpha\}$ and write $\beta = \sum_{\gamma\in\Delta} K_\gamma \cdot \gamma$ where $K_\gamma \geq 0$.

Note that β is not proportional to α. Thus $K_\gamma > 0$ for some $\gamma \neq \alpha$.

Then the coefficient of γ in $r_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ is also $K_\gamma > 0$.

Hence $r_\alpha(\beta)$ must still be in Φ^+ since it is a valid root.

We cannot have $r_\alpha(\beta) = \alpha$ since then $\beta = r_\alpha(r_\alpha(\beta)) = r_\alpha(\alpha) = -\alpha \notin \Phi^+$.

Thus r_α restricts to a map $\Phi^+ \setminus \{\alpha\} \to \Phi^+ \setminus \{\alpha\}$.

This map is a bijection since $\Phi^+ \setminus \{\alpha\}$ is a finite set and r_α is invertible. □

Corollary. Set $\delta = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta$. Then $r_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in \Delta$.

Lemma. Fix a sequence $\alpha_1, \alpha_2, \dots, \alpha_m \in \Delta$ of (not necessarily distinct) simple roots.

Write $r_i = r_{\alpha_i}$ and suppose $r_1 r_2 \cdots r_{m-1}(\alpha_m) \in \Phi^-$.

Then $\boxed{r_1 r_2 r_3 \cdots r_m = r_1 \cdots r_{s-1} r_{s+1} \cdots r_{m-1}}$ for some index $1 \le s \le m-1$.

Proof. Set $\beta_i = r_{i+1} r_{i+2} \cdots r_{m-1}(\alpha_m)$ with $\beta_{m-1} = \alpha_m$.

As $\beta_0 \in \Phi^-$ and $\beta_{m-1} \in \Delta \subset \Phi^+$ there is a smallest index s with $\beta_s \in \Phi^+$.

By definition $r_s(\beta_s) = \beta_{s-1}$ so $r_s(\beta_{s-1}) = \beta_s$ since $r_s^2 = 1$.

This means that $\beta_s \in \Phi^+$ and $r_s(\beta_s) \in \Phi^-$, so we must have $\beta_s = \alpha_s$ by the previous lemma.

Thus $r_{\beta_s} = r_{\alpha_s} = r_s$. Since $\sigma r_\alpha \sigma^{-1} = r_{\sigma(\alpha)}$, we deduce that

$$(r_{s+1} r_{s+2} \cdots r_{m-1}) r_m (r_{m-1} \cdots r_{s+2} r_{s+1}) = r_{r_{s+1} r_{s+2} \cdots r_{m-1}(\alpha_m)} = r_{\beta_s} = r_s.$$

The result now follows by substituting the left side of this equation for r_s in $r_1 r_2 r_3 \cdots r_m$. □

Corollary. Let $\sigma = r_{\alpha_1} \dots r_{\alpha_m}$ be an expression for $\sigma \in W$ with m as small as possible, where $\alpha_i \in \Delta$. Then we must have $\sigma(\alpha_m) \in \Phi^-$.

Recall: Φ is a root system with Weyl group W.

Proposition. Any given $\alpha \in \Phi$ belongs to some base of Φ.

Proof. The hyperplanes H_β for $\beta \in \Phi \setminus \{\pm\alpha\}$ are distinct from H_α.

Thus we can choose $\gamma \in H_\alpha$ with $\gamma \notin H_\beta$ for all $\beta \in \Phi \setminus \{\pm\alpha\}$ and then find a regular γ' close to γ with

$$(\gamma', \alpha) = \epsilon \quad \text{and} \quad (\gamma, \beta) > \epsilon$$

for all $\beta \in \Phi \setminus \{\pm\alpha\}$, for some arbitrarily small $\epsilon > 0$. Then we will have $\alpha \in \Delta(\gamma')$, □

Choose a simple system Δ for Φ.

Theorem. If Δ' is a simple system for Φ then there exists a unique element $\sigma \in W$ with $\sigma(\Delta') = \Delta$. Moreover, it holds that $W = \langle r_\alpha : \alpha \in \Delta \rangle$.

Recall that our initial definition of the Weyl group had more generators: $W = \langle r_\alpha : \alpha \in \Phi \rangle$.

Proof. Let $\widetilde{W} = \langle r_\alpha : \alpha \in \Delta \rangle \subseteq W$. We will show later that $\widetilde{W} = W$.

Let $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ and choose a regular $\gamma \in E$ along with $\sigma \in \widetilde{W}$ such that $(\sigma(\gamma), \delta)$ is maximal.

If α is simple then $r_\alpha\sigma \in \widetilde{W}$ so our maximality assumption implies that

$$(\sigma(\gamma), \delta) \geq (r_\alpha\sigma(\gamma), \delta) = (\sigma(\gamma), r_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha)$$

for all $\alpha \in \Delta$. Thus $(\sigma(\gamma), \alpha) \geq 0$ for all $\alpha \in \Delta$.

Equality never holds since γ is regular and $0 \neq (\gamma, \sigma^{-1}(\alpha)) = (\sigma(\gamma), \alpha)$.

Hence $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$.

If Δ' is a simple system then $\Delta' = \Delta(\gamma)$ for some regular $\gamma \in E$.

So if we choose $\sigma \in \widetilde{W}$ as above then $\Delta = \Delta(\sigma(\gamma)) = \sigma^{-1}(\Delta(\gamma)) = \sigma^{-1}(\Delta')$.

Thus for any simple system Δ' there is some $\sigma \in \widetilde{W} \subseteq W$ with $\sigma(\Delta') = \Delta$.

We now show that $\widetilde{W} = W$. It suffices to check that $r_\alpha \in \widetilde{W}$ for all $\alpha \in \Phi$.

Given $\alpha \in \Phi$, choose a simple system Δ' with $\alpha \in \Delta'$ and then choose $\sigma \in \widetilde{W}$ with $\sigma(\Delta') = \Delta$.

Set $\beta = \sigma(\alpha) \in \Delta$. Then $r_\beta \in \widetilde{W}$ so since $r_\beta = r_{\sigma(\alpha)} = \sigma r_\alpha \sigma^{-1}$ it also follows that $r_\alpha = \sigma^{-1} r_\beta \sigma \in \widetilde{W}$.

We conclude that $\widetilde{W} = W$.

Finally, we need to show that the $\sigma \in \widetilde{W} = W$ with $\sigma(\Delta') = \Delta$ is unique for a given simple system Δ'.

Showing this is where we use the technical lemma above.

It is enough to prove that if $\sigma \in W$ has $\sigma(\Delta) = \Delta$ then $\sigma = 1$.

Assume $\sigma(\Delta) = \Delta$ and write $\sigma = r_1 r_2 \cdots r_m$ where $r_i = r_{\alpha_i}$ for simple roots $\alpha_1, \alpha_2, \ldots, \alpha_m \in \Delta$.

Let m be as small as possible.

If $\sigma \neq 1$ then $m > 0$ so by the corollary above we have $\sigma(\alpha_m) \in \Phi^-$ so $\sigma(\Delta) \neq \Delta \subseteq \Phi^+$.

Thus the only way to have $\sigma(\Delta) = \Delta$ is if $m = 0$ and then $\sigma = 1$. □

Fix an ordering $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ of the roots in Δ.

Here $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and $\alpha_i \neq \alpha_j$ for $i \neq j$.

Abbreviate by writing $r_j = r_{\alpha_j}$.

We call a minimal-length expression $\sigma = r_{\alpha_1} r_{\alpha_2} \ldots r_{\alpha_m}$ a *reduced expression* for $\sigma \in W$.

Set $\ell(\sigma) = m$ and call this the *length* of σ.

Proposition. If $\sigma \in W$ then $\ell(\sigma) = |\{\alpha \in \Phi^+ : \sigma(\alpha) \in \Phi^-\}|$. In particular, $\ell(r_\alpha) = 1$ for all $\alpha \in \Delta$.

Proof. Use induction + our earlier lemmas; see the textbook for a detailed argument. □