MATH 5143 - Lecture 13

Last time: bases/simple systems of root systems

E is a real vector space with a symmetric, positive definite, bilinear form $(\cdot,\cdot)$

A nonempty finite subset $\Phi \subseteq E \setminus \{0\}$ is a <u>root system</u>

if (a) $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ $\forall \alpha \in \Phi$

(b) $r_\alpha(\Phi) = \Phi$ $\forall \alpha \in \Phi$ where $r_\alpha : x \mapsto x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$

(c) $2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ $\forall \alpha, \beta \in \Phi$

(d) E is spanned by Φ

The Weyl group of Φ is then $W \overset{\text{def}}{=} \langle r_\alpha \mid \alpha \in \Phi \rangle \subseteq GL(E)$

For $0 \neq \alpha \in E$ let $H_\alpha = \{x \in E \mid (x, \alpha) = 0\}$

If Φ is any finite set in $E \setminus \{0\}$ then

$$E \setminus \bigcup_{\alpha \in \Phi} H_\alpha \text{ is nonempty.}$$ [Easy to visualize if $E = \mathbb{R}^2$]

So it is possible to choose some $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.

For this γ we have $(\gamma, \alpha) \neq 0 \ \forall \alpha \in \Phi$ so can set

$$\Phi^+(\gamma) \stackrel{\text{def}}{=} \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\} \text{ and } \Phi^-(\gamma) \stackrel{\text{def}}{=} -\Phi^+(\gamma)$$

Define $\Delta(\gamma) = \{\alpha \in \Phi^+(\gamma) \mid$ there are no elements $\beta_1, \beta_2 \in \Phi^+(\gamma)$ with $\alpha = \beta_1 + \beta_2\}$

Thm If Φ is a root system then the set $\Delta(\gamma)$ is a **base** (or **simple system**) for Φ, meaning that

union almost disjoint, but both sets contain 0

$$\Phi \subseteq \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta(\gamma)\} \cup \mathbb{Z}_{\leq 0}\text{-span}\{\alpha \in \Delta(\gamma)\}$$

and that $\Delta(\gamma)$ is a basis for E. Moreover, every base of Φ arises from this construction as $\Delta(\gamma)$ for some $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.

Given a base $\Delta \subseteq \Phi$, call each $\alpha \in \Delta$ a **simple root** and each $\alpha \in \Phi^{+/-}$ a **positive** / **negative** root.

Fix a base Δ for Φ from now on. Some facts:

1. If $\alpha \in \Delta$ then $r_\alpha(\alpha) = -\alpha$ and $r_\alpha(\Phi^+ \setminus [\alpha]) = \Phi^+ \setminus [\alpha]$

2. $W \overset{\text{def}}{=} \langle r_\alpha \mid \alpha \in \Phi \rangle = \langle r_\alpha \mid \alpha \in \Phi^+ \rangle = \langle r_\alpha \mid \alpha \in \Delta \rangle$

   obvious since $\Phi = \Phi^+ \sqcup \Phi^-$ as $r_\alpha = r_{-\alpha}$

   nontrivial and useful

3. If $\beta \in \Phi$ then there is some base of Φ containing β and there is some $w \in W$ with $w(\beta) \in \Delta$.

4. If Δ' is another base, then there is a unique $w \in W$ with $w(\Delta) = \Delta'$.

Claim For a root system Φ with base Δ, the following are equivalent:

ⓐ we can write $\Phi = \Phi_1 \cup \Phi_2$ for some nonempty disjoint subsets Φ_i with $(\alpha, \beta) = 0 \ \forall \begin{smallmatrix} \alpha \in \Phi_1 \\ \beta \in \Phi_2 \end{smallmatrix}$

ⓑ we can write $\Delta = \Delta_1 \cup \Delta_2$ for some nonempty disjoint sets Δ_i with $(\alpha, \beta) = 0 \ \forall \begin{smallmatrix} \alpha \in \Delta_1 \\ \beta \in \Delta_2 \end{smallmatrix}$

[Φ is <u>reducible</u> in these cases]

Clearly if these properties hold, and $E_i \overset{\text{def}}{=} \mathbb{R}\text{-span}[\alpha \in \Delta_i]$, then $(\cdot,\cdot)$ restricts to a positive definite form on each E_i and $E = E_1 \oplus E_2$ and each Φ_i is a root system in E_i with Δ_i as a base

Proof of claim ⓐ $\Rightarrow$ ⓑ since we can just set $\Delta_i = \Delta \cap \Phi_i$ for $i = 1, 2$. The harder direction is to show that ⓑ $\Rightarrow$ ⓐ. For this, given $\Delta = \Delta_1 \sqcup \Delta_2$ let $\Phi_i^+ = \mathbb{Z}_{\geq 0}\text{-span}[\alpha \in \Delta_i] \cap \Phi$.

Let $\Phi_i^- = -\Phi_i^+$ and $\Phi_i = \Phi_i^+ \sqcup \Phi_i^-$.

Then $\Phi_1 \perp \Phi_2$ since $\Delta_1 \perp \Delta_2$. Why does $\Phi = \Phi_1 \cup \Phi_2$?

Suffices to show $\Phi^+ \overset{\text{def}}{=} \mathbb{Z}_{\geq 0}\text{-span}[\alpha \in \Delta] \cap \Phi$ is $\Phi_1^+ \sqcup \Phi_2^+$.

This holds since if $\alpha \in \Phi_1^+$ and $\beta \in \Phi_2^+$ then $r_\alpha(\alpha + \beta) = -\alpha + r_\alpha(\beta)$

$$= -\alpha + \Big(\beta - \underbrace{\frac{2(\beta,\alpha)}{(\alpha,\alpha)}}_{=0 \text{ as } \Phi_1 \perp \Phi_2}\alpha\Big) = \underbrace{\beta - \alpha}_{\text{involves coeffs of both signs when expanded in terms of } \Delta} \notin \Phi \Longrightarrow \alpha + \beta \notin \Phi. \quad \square$$

All of this extends from two to k factors as follows:

Prop There is a maximal partition $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_k$ into nonempty pairwise disjoint and orthogonal subsets, which is unique up to permutation of indices, and if $E_i \overset{\text{def}}{=} \mathbb{R}\text{-span}[\alpha \in \Delta_i]$ and $\Phi_i \overset{\text{def}}{=} \Phi \cap E_i$ then $E = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ and each Φ_i is is a root system in E_i with base Δ_i and $\Phi = \Phi_1 \sqcup \Phi_2 \sqcup \cdots \sqcup \Phi_k$

We call the root systems Φ_i the irreducible components of Φ.
The prop. shows that Φ is det'd up to $\cong$ by these components

Note: Φ is irreducible iff $k=1$ in the prop.

Pf. The only part that is not clear is claim that $\Phi = \Phi_1 \sqcup \Phi_2 \sqcup \cdots \sqcup \Phi_k$.
To show this, consider some $\gamma \in \Phi$. Then there is $w \in W$ with $w(\gamma) \in \Delta$, so γ is in W-orbit of an element of some Δ_i. But orthogonality + $W = \langle r_\alpha \mid \alpha \in \Delta \rangle$ means that W preserves the subspace E_i so $\gamma \in \Phi_i$. $\square$

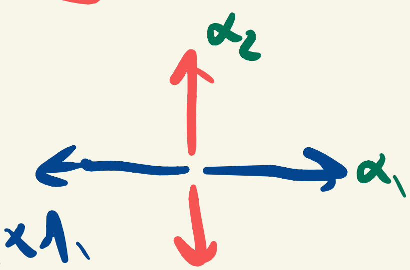
Invariants of root systems: the Cartan matrix, the Coxeter graph, and the Dynkin diagram of Φ.

Fix an ordering $\alpha_1\ \alpha_2\ \alpha_3 \dots \alpha_\ell$ of the simple roots in our fixed base $\Delta \subseteq \Phi$.

<u>Def</u> (with respect to this ordering) the <u>Cartan matrix</u> of Φ is the $\ell \times \ell$ matrix $\left[\langle \alpha_i, \alpha_j \rangle\right]_{1 \le i,j \le \ell}$ where

$$\langle \alpha, \beta \rangle \overset{\text{def}}{=} 2(\alpha,\beta)/(\beta,\beta) \in \mathbb{Z}.$$

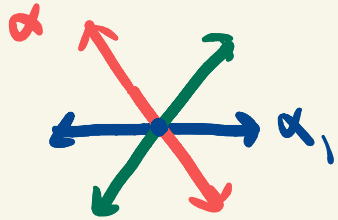
<u>Ex</u>. Cartan matrices for root systems in $\mathbb{R}^2$.



$\Phi_{A_1 \times A_1}$

Cartan matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ as $(\alpha_1, \alpha_2) = 0$

Φ_{A_2}



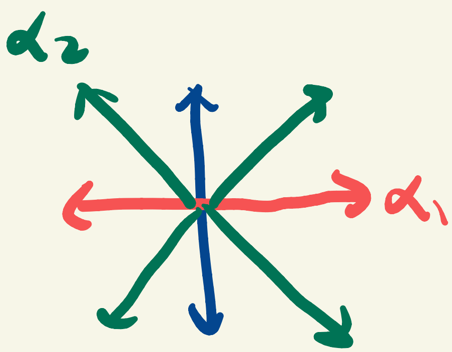
Then $(\alpha_1, \alpha_2) = \|\alpha_1\| \|\alpha_2\| \cos(2\pi/3)$

and $\|\alpha_1\| = \|\alpha_2\|$ so we have

$$\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle = 2\cos(2\pi/3) = -1$$

Cartan matrix $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Φ_{B_2}



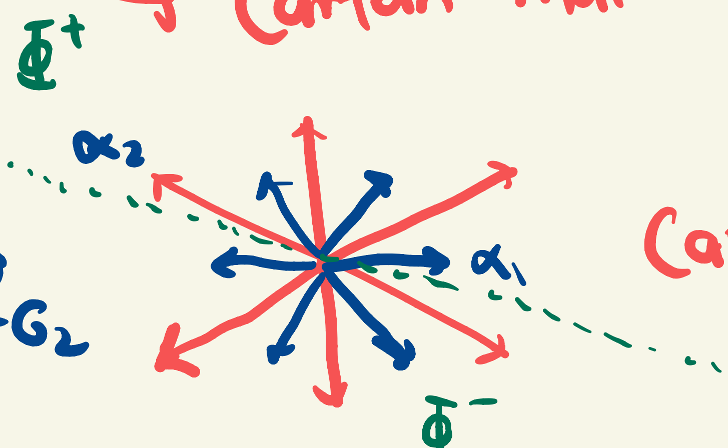
$(\alpha_1, \alpha_1) = 1$

$(\alpha_2, \alpha_2) = 2$

$(\alpha_1, \alpha_2) = \sqrt{1}\sqrt{2} \cdot \cos \frac{3\pi}{2} = -1$

Cartan matrix $= \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$ or $\langle \alpha, \beta \rangle = \dfrac{2(\alpha, \beta)}{(\beta, \beta)}$

Φ_{G_2}



Cartan matrix works out to $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Prop. The Cartan matrix (up to reordering of rows / cols) determines Φ (up to isomorphism). More precisely, if there is another root system $\Phi' \subseteq E'$ with ordered base Δ' and there is a bijection $f: \Delta \to \Delta'$ such that

$$\langle \alpha, \beta \rangle = \langle f(\alpha), f(\beta) \rangle \quad \forall \alpha, \beta \in \Delta$$

then the unique linear map $E \to E'$ extending f is a root system isomorphism $\Phi \xrightarrow{\sim} \Phi'$. In particular, the linear extension of f has $\langle \alpha, \beta \rangle = \langle f(\alpha), f(\beta) \rangle \; \forall \alpha, \beta \in \Phi$.

<u>Pf</u> The linear extension $f : E \to E'$ is invertible since Δ, Δ' are bases.

For $\alpha \in \Delta$, it holds that $r_{f(\alpha)} = f \circ r_\alpha \circ f^{-1}$.

Hence the Weyl group W' of Φ is exactly

$$\{ f \circ w \circ f^{-1} \mid w \in W \}.$$

Each $\beta \in \Phi$ has $\beta = w(\alpha)$ for some $w \in W$, $\alpha \in \Delta$.

so $f(\beta) = f \circ w(\alpha) = \underbrace{f \circ w \circ f^{-1}}_{\in W'} (\underbrace{f(\alpha)}_{\in \Delta'}) \in \Phi'$.

imply

Similar argument shows that $f^{-1}(\beta) \in \Phi$ $\forall \beta \in \Phi'$ so we can conclude that f is a bijection $\Phi \to \Phi'$.

Finally observe for $\alpha, \beta \in \Phi$ that

$$r_{f(\alpha)}(f(\beta)) = f \circ r_\alpha \circ f^{-1}(f(\beta)) = f(r_\alpha(\beta))$$

def

$$= f(\beta) - \langle f(\beta), f(\alpha)\rangle f(\alpha) \qquad = f(\beta) - \langle \beta, \alpha\rangle f(\alpha)$$

So we must have $\langle \beta, \alpha \rangle = \langle f(\beta), f(\alpha)\rangle$.

[Checking that $r_{f(\alpha)} = f \circ r_\alpha \circ f^{-1}$ for any $\alpha \in \Phi$ follows from case when $\alpha \in \Delta \longrightarrow$ exercise.]

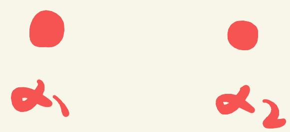
$\square$

Coxeter graph of a root system Φ with base Δ: this is the undirected graph with vertices labeled by the elements of Δ and with exactly $\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle = \dfrac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$ edges between vertices α and β.

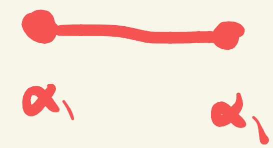
(is in $\mathbb{Z}_{\geq 0}$)

Examples of Coxeter graphs for $\Phi \subseteq \mathbb{R}^2$:

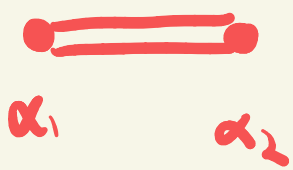
$\Phi_{A_1 \times A_1}$:



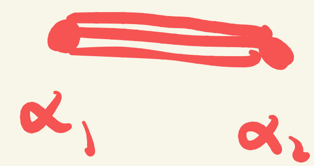
Φ_{A_2}:



Φ_{B_2}:



Φ_{G_2}:



the # of edges between α_i and α_j is the product of entries (i,j) and (j,i) of Cartan matrix.

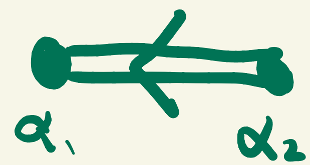
If all roots have same length (eg for Φ_{A_2}) then $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$

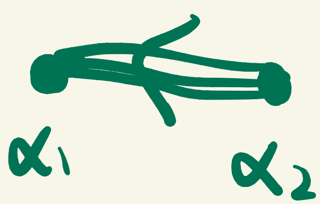
If roots have different lengths then we need a little extra information to recover the Cartan matrix from the Coxeter graph.

Define the <u>Dynkin diagram</u> of Φ by taking the Coxeter diagram and adding an arrow from longer root to shorter root to each double or triple edge.

$\Phi_{A_1 \times A_1}$ and Φ_{A_2}: Coxeter graph = Dynkin diagram

Dynkin diagram of

- Φ_{B_2} is  $\|\alpha_2\| > \|\alpha_1\|$

- Φ_{G_2} is 

Dynkin diagram determines the Cartan matrix

$\Rightarrow$ Cor. The Dynkin diagram of Φ determines Φ up to $\cong$

Moreover, the irreducible components of Φ correspond to the connected components of the Dynkin diagram, and so Φ is irreducible iff the Dynkin diagram is connected.

Next: classification results and constructions