MATH 5143 - Lecture 14

Last time

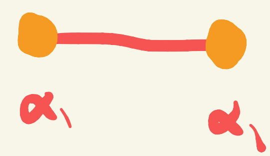
Let Φ be a root system w/ simple system Δ

Coxeter graph of Φ is undirected graph with vertices labeled by the elements of Δ and with exactly $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle \in \mathbb{Z}_{\geq 0}$ edges between vertices α and β.

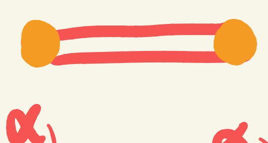
Examples of Coxeter graphs for $\Phi \subseteq \mathbb{R}^2$:

$\Phi_{A_1 \times A_1}$:

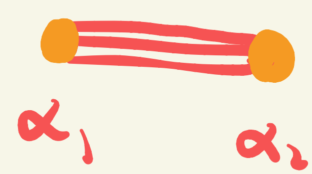
Φ_{A_2} :



Φ_{B_2} :



Φ_{G_2} :

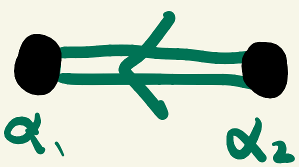


Define the <u>Dynkin diagram</u> of Φ by taking the Coxeter diagram and adding an arrow from longer root to shorter root to each double or triple edge.

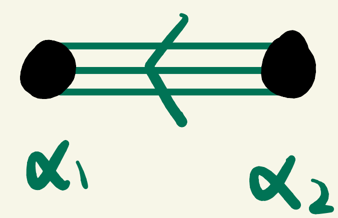
<u>Examples</u>

$\Phi_{A_1 \times A_1}$ and Φ_{A_2}: Dynkin diagram $=$ Coxeter graph

Φ_{B_2}:



Φ_{G_2}:



<u>Thm</u> The Dynkin diagram determines Φ up to $\cong$.

Some constructions

Prop Let $E = \{ v \in \mathbb{R}^{n+1} \mid v_1 + v_2 + v_3 + \cdots + v_{n+1} = 0 \} \cong \mathbb{R}^n$

is orthogonal complement of $\mathbb{R}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n+1})$

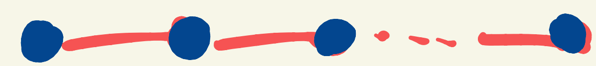
so inherits nondeg. form from $\mathbb{R}^{n+1}$

Write $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}$ for standard basis of $\mathbb{R}^{n+1}$.

Define $\Phi_{A_n} = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n,\ i \neq j \}$

Then Φ_{A_n} is a root system with base $\Delta_{A_n} = \{ \varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \ldots, n \}$

and Dynkin diagram



Also, $r_{\varepsilon_i - \varepsilon_{i+1}}\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} - (v_i - v_{i+1})(\varepsilon_i - \varepsilon_{i+1}) = \begin{bmatrix} v_1 \\ \vdots \\ v_{i+1} \\ v_i \\ \vdots \\ v_n \end{bmatrix}$

so it follows that Weyl group $W_{A_n} \cong S_{n+1}$ (symmetric group of $1, 2, \ldots, n+1$)

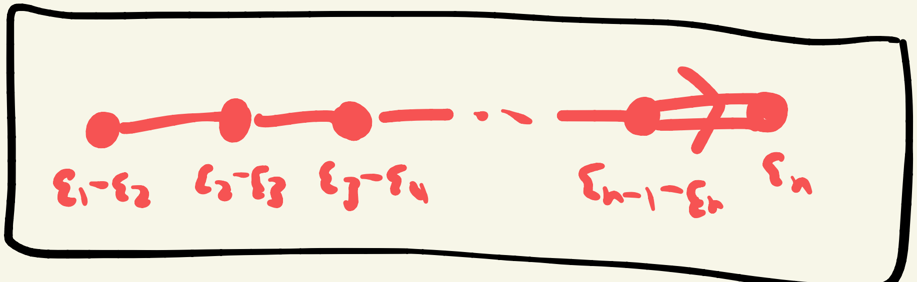
Pf All straightforward calculations. $\square$

Prop Let $\Phi_{B_n} \subseteq \mathbb{R}^n$ be set of $2n + 4\binom{n}{2}$ vectors

$$\{\pm \varepsilon_i \mid i = 1, 2, \dots, n\} \sqcup \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

Then Φ_{B_n} is a root system with base

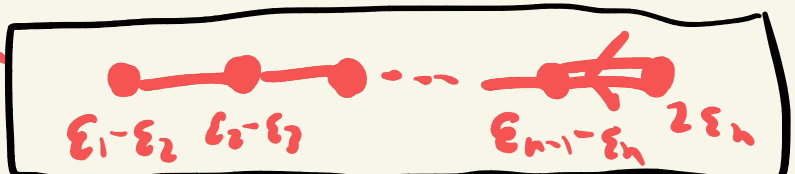
$$\Delta_{B_n} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$$

and Dynkin diagram 

The Weyl group $W_{B_n} \cong$ (signed $n \times n$ permutation matrices)

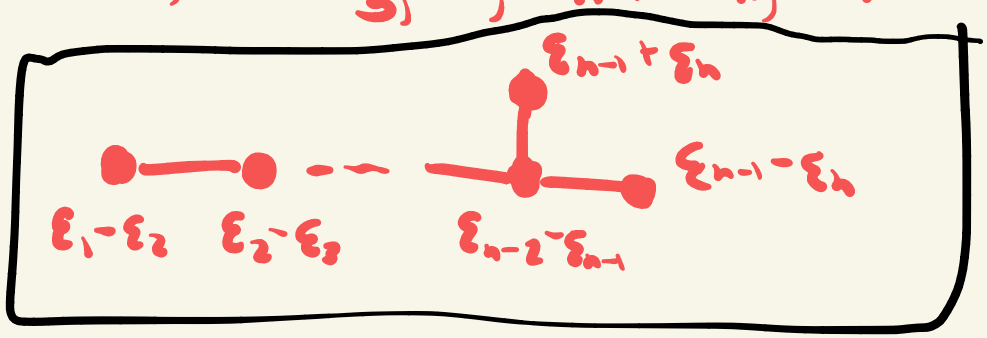
Prop Let $\Phi_{C_n} = \{\pm 2\varepsilon_i \mid i = 1, 2, \dots, n\} \sqcup \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$

Then Φ_{C_n} is a root system with base $\Delta_{C_n} = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$

and Dynkin diagram 

Also: $W_{C_n} = W_{B_n}$.

Prop Finally let $\Phi_{D_n} \subseteq \mathbb{R}^n$ be set of $4\binom{n}{2}$ vectors $\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i<j \le n\}$. Then Φ_{D_n} is a root system with base $\Delta_{D_n} = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \dots, \varepsilon_{n-1}-\varepsilon_n, \varepsilon_{n-1}+\varepsilon_n\}$ and Dynkin diagram



The Weyl group W_{D_n} is an index two normal subgroup of $W_{B_n} = W_{C_n}$

(subgroup of even signed $n \times n$ permut. matrices)

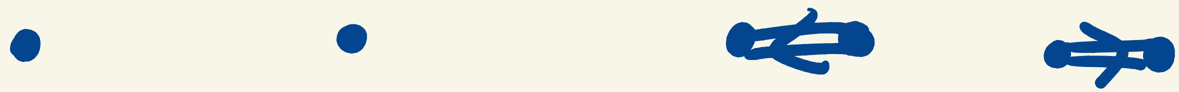
even # of -1 entries

Φ_{A_n} is irreducible $\forall\, n \geq 1$ (Dynkin diagram is connected)

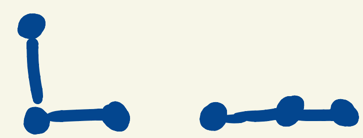
$\Phi_{B_1} \cong \Phi_{A_1}$ as Dynkin diagram is just an isolated vertex

So we only consider Φ_{B_n} for $n \geq 2$

$\Phi_{C_1} \cong \Phi_{B_1}$ and $\Phi_{C_2} \cong \Phi_{B_2}$ since Dynkin diagrams are isomorphic.



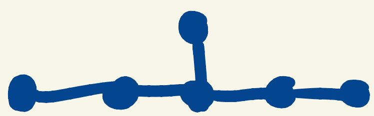
so we only consider Φ_{C_n} for $n \geq 3$.

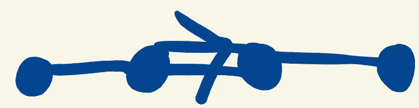


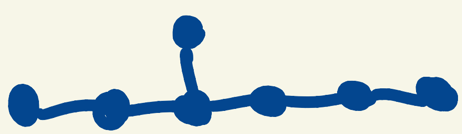
$\Phi_{D_1} \cong \Phi_{A_1}$, $\Phi_{D_2} \cong \Phi_{A_1 \times A_1}$ (not irreducible) $\Phi_{D_3} \cong \Phi_{A_3}$

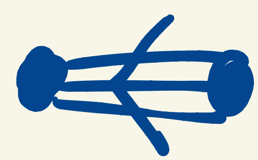
so we only consider Φ_{D_n} for $n \geq 4$.

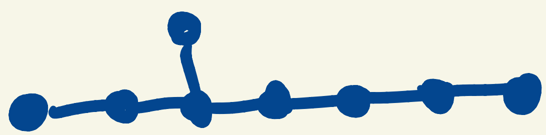
Thm Suppose Φ is an irreducible root system. Then the Dynkin diagram of Φ is either isomorphic to the Dynkin diagram of Φ_{A_n} (some $n \geq 1$), Φ_{B_n} (some $n \geq 2$), Φ_{C_n} (some $n \geq 3$), Φ_{D_n} (some $n \geq 4$), or to one of 5 "exceptional" diagrams:

E_6 

F_4 

E_7 

G_2 

E_8 

Moreover, each of these exceptional diagrams does arise as the Dynkin diagram of an (irreducible) root system.

Proof that every irreducible root system has one of these Dynkin diagrams $\longrightarrow$ See the relevant sections of textbook if interested

Constructions for E_n, F_4, G_2

Φ_{G_2}: we've already seen

$$\Phi_{F_4} \overset{\text{def}}{=} \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le 4\} \sqcup \left\{\tfrac{1}{2}(a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3 + a_4\varepsilon_4) \;\middle|\; a_1, a_2, a_3, a_4 \in \{\pm 1\}\right\}$$

$$\sqcup \{\pm\varepsilon_i \mid 1 \le i \le 4\} \subseteq \mathbb{R}^4$$

has 48 elements, Weyl group has size 1152.

has base $\{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$

Suffices to construct Φ_{E_8} as Φ_{E_6}, Φ_{E_7} can then be realized as subsystems. We can construct $\Phi_{E_8} \subseteq \mathbb{R}^8$ as the set of 240 vectors of the form

$$\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 8\} \sqcup \left\{ \tfrac{1}{2}(a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_8\varepsilon_8) \;\; \begin{array}{l} a_1, a_2, \ldots, a_8 \in \{\pm 1\} \text{ with} \\ a_1 \cdot a_2 \cdot a_3 \cdots a_8 = +1 \end{array} \right\}$$

This is a root system with base

$$\Delta_{E_8} = \left\{ \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8),\ \varepsilon_1 + \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_5,\ \varepsilon_5 - \varepsilon_6,\ \varepsilon_6 - \varepsilon_7,\ \varepsilon_7 - \varepsilon_8 \right\}$$

Isomorphism and conjugacy theorems

Recall that if L is a semisimple Lie algebra (over an alg. closed, char. zero field $\mathbb{F}$), and $H \subseteq L$ is a maximal toral subalgebra then there is a finite set $\Phi \subseteq H^* \setminus \{0\}$ with $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ where

$$L_\alpha \overset{\text{def}}{=} \{x \in L \mid [h,x] = \alpha(h)x \ \forall h \in\} \neq 0 \text{ for } \alpha \in \Phi.$$

The set Φ is a root system in $E = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}\text{-span}\{\alpha \in \Phi\}$ with the bilinear form on H^* dual to the killing form of L restricted to H.

Prop If L is simple then Φ is irreducible

Pf If $\Phi = \Phi_1 \cup \Phi_2$ were reducible (with Φ_1, Φ_2 nonempty and orthogonal)

and $\alpha \in \Phi_1$, $\beta \in \Phi_2$, then $\alpha+\beta$ is neither in

Φ_1 (since $(\beta, \alpha+\beta) = (\beta, \beta) \neq 0$) nor in Φ_2

(since $(\alpha, \alpha+\beta) = (\alpha, \alpha) \neq 0$) so $\alpha+\beta \notin \Phi$

and it follows that the subalgebra of L generated

by L_α for $\alpha \in \Phi_1$ is a proper nonzero ideal.

(since $[L_\alpha, L_\beta] = 0 \ \forall \alpha \in \Phi_1, \beta \in \Phi_2$) $\square$

Prop If $L = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ is the decomposition of L into simple ideals then $H_i \stackrel{\text{def}}{=} H \cap L_i$ is a maximal toral subalgebra of L_i and the irreducible root system Φ_i determined by $H_i \subseteq L_i$ may be viewed as a subsystem of Φ relative to which

$$\Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_n$$

is the decomposition of Φ into irreducible components.

Pf See discussion in §14.1 of textbook. $\square$

Thm Suppose L' is another semisimple Lie algebra with a maximal toral subalgebra H' and root system Φ'. Suppose there exists a root system isomorphism $f : \Phi \to \Phi'$.

Extend f to a vector space isomorphism $f : H \xrightarrow{\sim} H'$ by setting $f(t_\alpha) = t'_{f(\alpha)}$ where for $\alpha \in \Phi$, $\alpha' \in \Phi'$, $t_\alpha \in H$ and $t'_{\alpha'} \in H'$ are the elements with $\begin{cases} \kappa(t_\alpha, h) = \alpha(h) \\ \kappa(t'_{\alpha'}, h') = \alpha'(h') \end{cases}$

Choose a base $\Delta \subseteq \Phi$ along with isomorphisms between the 1-dim root spaces $L_\alpha \xrightarrow{\sim} L'_{f(\alpha)}$ for $\alpha \in \Delta$. Then there is a unique Lie algebra isomorphism $L \xrightarrow{\sim} L'$ extending $f : H \to H'$ and these chosen isomorphisms.

This theorem does require some proof → see textbook.

Proof in §14 of textbook does not establish existence of a semisimple Lie algebra corresponding to any Dynkin diagram.

Existence of this is shown in §18, we may discuss in a few lectures. (For classical type ABCD, just use the classical algebras $\mathfrak{sl}_n$, $\mathfrak{o}_n$, $\mathfrak{sp}_n$ — main issue is about exceptional types)