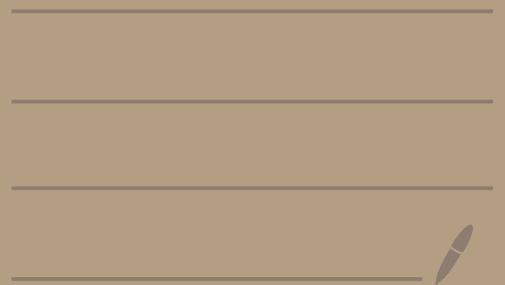


MATH 5143 - Lecture 15



Last time: Cartan matrix
 Coxeter diagrams
 Dynkin diagrams } Classification of
 irreducible root
 systems (and
 by extension all simple
 Lie algebras)

A root system in a real vector space E

with a symmetric, positive def. bilinear form, is a nonempty

finite set $\Phi \subset E \setminus \{0\}$ which spans E , which has

$\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\} \forall \alpha \in \Phi$, which has $r_\alpha(\Phi) = \Phi \forall \alpha \in \Phi$,

and which has $2(\beta, \alpha) / (\alpha, \alpha) \in \mathbb{Z} \forall \alpha, \beta \in \Phi$.

- * Call the integers $2(\beta, \alpha) / (\alpha, \alpha)$ the Cartan numbers
- * An isomorphism between root systems of Φ is a bijection preserving Cartan numbers
- * A root system is irreducible if it cannot be partitioned into disjoint nonempty orthogonal subsets

Key point: every root system has a unique decomposition as a disjoint union of irreducible subsystems (where a subset of Φ is viewed as a root system in the subspace of E that it spans)

The isomorphism classes of irreducible root systems belong to 7 families:

A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$)

E_6, E_7, E_8 , F_4 , and G_2

Last time: we saw constructions in types ABCD

Choose a simple system / base Δ in a root system Φ

Order Δ as
 $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$.

↳ recall this is a basis for E such that we never need to mix signs of coefficients when expressing roots in terms of Δ .

Then the Cartan matrix of Φ is $[2(\alpha_i, \alpha_j) / (\alpha_j, \alpha_j)]_{1 \leq i, j \leq n}$

The Coxeter diagram of Φ is the graph

with vertex set Δ and with $\frac{4(\alpha_i, \alpha_j)(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} \in \mathbb{Z}_{\geq 0}$ edges

between α_i and α_j for each $\alpha_i \neq \alpha_j$ in Δ .

The Dynkin diagram of Φ is formed from the

Coxeter diagram by orienting edges $\alpha = \beta$

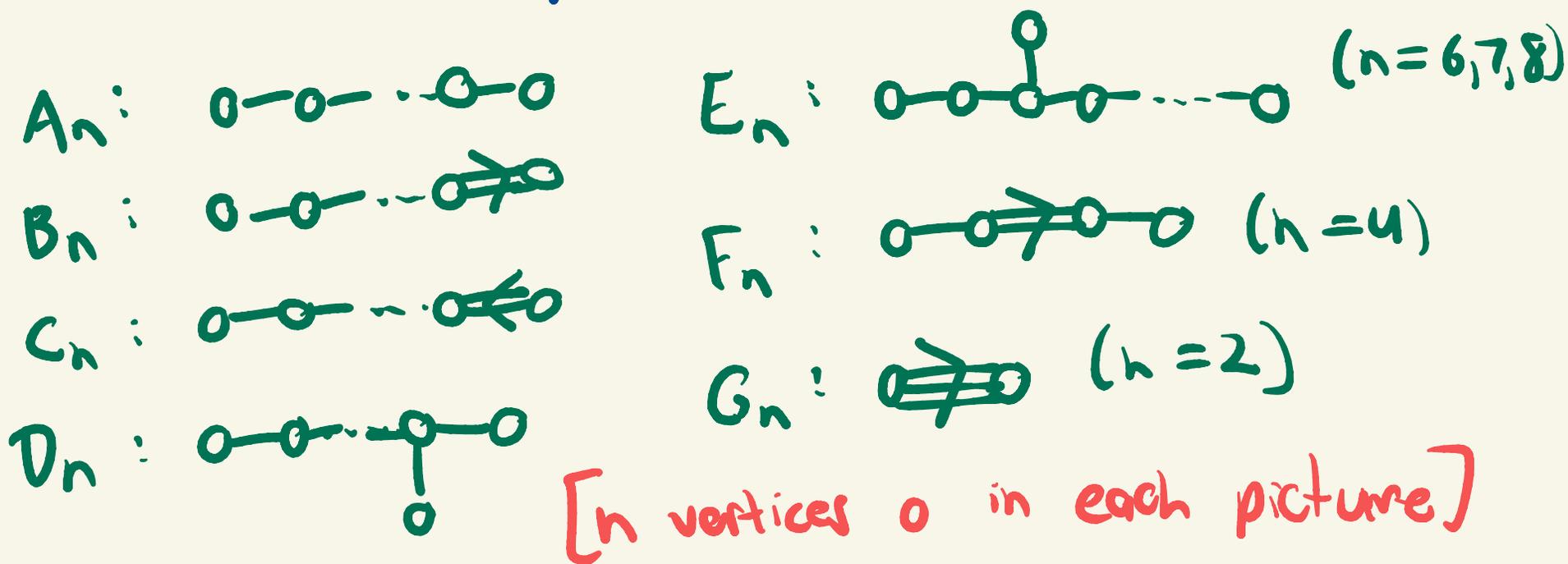
and $\alpha \equiv \beta$ to be $\alpha \rightrightarrows \beta$ and $\alpha \rightrightarrows \rightrightarrows \beta$

if $\|\alpha\| > \|\beta\|$. [Whenever a double or triple edge occurs, it is between roots of different lengths]

Thm The Cartan matrix and Dynkin diagram [up to arbitrary relabelling of indices / vertices] uniquely

determine the isomorphism class of Φ . Also, Φ is irreducible iff the Dynkin diagram is connected.

The graphs that occur as Dynkin diagrams of irreducible root systems are precisely



Back to semisimple Lie algebras suppose L is

a finite-dimensional Lie algebra over an algebraically closed field \mathbb{F} of char. zero. Assume L is semisimple [so

L is direct sum of simple Lie algebras / has no solvable ideals]

"consists of all semisimple elems" \leftarrow L has no proper nonzero ideals and is non-abelian"

Then: for any maximal toral subalgebra $\mathfrak{H} \subseteq L$ there is a (unique) decomposition $L = \mathfrak{H} \oplus \bigoplus_{\alpha \in \Phi} L_{\alpha}$ for a certain subset $\Phi \subset \mathfrak{H}^*$,

where $L_{\alpha} \stackrel{\text{def}}{=} \{x \in L \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{H}\}$.

The vector space \mathfrak{h}^* has a nondegenerate symmetric form dual to the restriction to \mathfrak{h} of the Killing form

$$\chi(X, Y) \stackrel{\text{def}}{=} \text{trace}(\text{ad} X \text{ ad} Y) \text{ for } X, Y \in L.$$

The set Φ is a root system in \mathfrak{h}^* .

Thm L is simple iff Φ is irreducible.

Also, two semisimple Lie algebras are isomorphic if and only if their root systems are isomorphic.

one thing we haven't seen (for exceptional types):

for each irreducible root system there does exist a simple Lie algebra with that root system as its Φ .

Digression — Cartan subalgebras

Quick overview, no proofs

Def A Cartan subalgebra of a Lie algebra L is a nilpotent subalgebra $H \subseteq L$ with $H = N_L(H)$.

Here $N_L(H) \stackrel{\text{def}}{=} \{x \in L \mid [x, h] \in H \ \forall h \in H\}$

Thm If L is semisimple and defined over an alg.-closed field \mathbb{F} with $\text{char}(\mathbb{F}) = 0$, then a subalgebra $H \subseteq L$ is a maximal toral subalgebra iff H is a Cartan subalg.

If $\text{char}(\mathbb{F}) > 0$, then these are different, however.

Def A Borel subalgebra of a Lie algebra L is a maximal solvable subalgebra

Thm If \mathfrak{B}_1 and \mathfrak{B}_2 are two Borel subalgebras of a Lie algebra L , then there is an automorphism $f \in \text{Aut}(L)$ with $f(\mathfrak{B}_1) = \mathfrak{B}_2$. Moreover, the same fact holds if \mathfrak{B}_1 and \mathfrak{B}_2 are two Cartan subalgebras.

Ex If $L = \text{SL}_n(\mathbb{F})$ then two Borel subalgebras are $\mathfrak{B}_1 =$ upper Δ -matrices and $\mathfrak{B}_2 =$ lower- Δ matrices.

We have $f(\mathfrak{B}_1) = \mathfrak{B}_2$ for $f(x) = -x^T$.

Textbook proves stronger fact that $f \in \text{Aut}(L)$ can be chosen in a subgroup $\mathcal{E}(L) \subseteq \text{Aut}(L)$ generated by $\exp(\text{ad } x)$ for $x \in L$ that are **strongly ad-nilpotent** in a certain sense.

Related important fact:

Cor In a general Lie algebra, all Cartan subalgebras \mathfrak{h} are isomorphic.

New topic today:

universal enveloping algebras.

The following constructions pertain to arbitrary Lie algebras over any field \mathbb{F} . The main idea is to construct from

a Lie algebra L an associative unital algebra $U(L)$

"as freely as possible" subject to the commutation

relations of L . That is, we want to build the

"most general possible" algebra $U(L) \supseteq L$ that has

$$X \cdot Y - Y \cdot X = [X, Y] \quad \forall X, Y \in L.$$

An associative unital algebra is just a vector space A with an associative bilinear multiplication operation and a compatible unit $1 \in A$.

Def An enveloping algebra of a given Lie algebra L

is a pair (A, ϕ) where A is an associative unital algebra and $\phi: L \rightarrow A$ is a linear map, such that

$$\phi([X, Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X) \quad \forall X, Y \in L.$$

Ex If $L \subseteq \mathfrak{gl}(V)$ for a vector space V then

$\mathfrak{gl}(V)$ is an enveloping algebra wrt to obvious inclusion $\phi: L \hookrightarrow \mathfrak{gl}(V)$

A morphism of enveloping algebras $f: (A_1, \phi_1) \rightarrow (A_2, \phi_2)$

is an algebra homomorphism $f: A_1 \rightarrow A_2$ such that

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ \phi_1 \uparrow & & \downarrow \phi_2 \\ & L & \end{array} \text{ commutes}$$

a linear map sending units to units
commuting with multiplication

Def A universal enveloping algebra of L is

an initial object in the category of enveloping algebras

for L : that is, an enveloping algebra (U, i) such that

if (A, ϕ) is any enveloping algebra for L then there

is a unique morphism $(U, i) \rightarrow (A, \phi)$

Prop If (U_1, i_1) and (U_2, i_2) are both universal enveloping algebras

for L then there is a unique isomorphism $(U_1, i_1) \xrightarrow{\sim} (U_2, i_2)$

Pf By def., there are unique morphisms

$f: (U_1, i_1) \rightarrow (U_2, i_2)$ and $g: (U_2, i_2) \rightarrow (U_1, i_1)$.

↳ morphism with a two-sided inverse, i.e. an algebra isomorphism commuting with relevant diagrams.

But the identity morphism is the only morphism $(U_j, i_j) \rightarrow (U_j, i_j)$. So $f \circ g = \text{id}$
 $g \circ f = \text{id} \quad \square$

So there is at most one universal enveloping algebra of L .
(up to unique isomorphism). More involved:

Thm Any Lie algebra L has a universal enveloping algebra. [This is always infinite-dimensional if $L \neq 0$].

The proof requires a short digression on tensor algebras.

Let V be a finite-dimensional vector space over a field \mathbb{F} .

Define $T^0 V = \mathbb{F}$

$$T^3 V = V \otimes V \otimes V$$

$$T^1 V = V$$

\vdots

$$T^2 V = V \otimes V$$

$$T^n V = V \otimes V \otimes \dots \otimes V \text{ (n factors)}$$

Let $T(V) = \bigoplus_{n \geq 0} T^n V$. This is a vector space whose elements

are finite linear combinations of tensors $v_1 \otimes v_2 \otimes \dots \otimes v_n$ for any $n \geq 0$, any $v_i \in V$

We make $T(V)$ into an associative unital algebra with unit

$1 \in \mathbb{F} = T^0 V \subset T(V)$ by setting

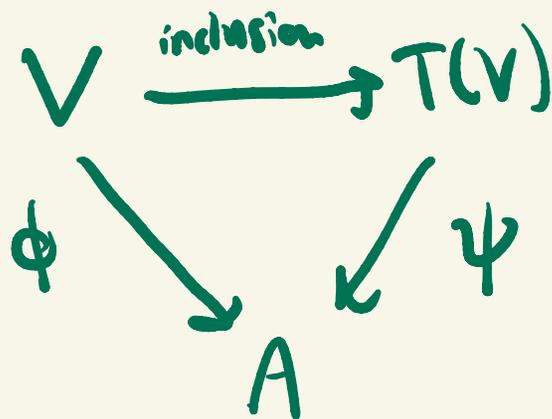
$$(v_1 \otimes \dots \otimes v_k)(w_1 \otimes \dots \otimes w_\ell) \stackrel{\text{def}}{=} v_1 \otimes \dots \otimes v_k \otimes w_1 \otimes \dots \otimes w_\ell$$

and extending by linearity, for $v_i, w_j \in V$. The resulting structure is the tensor algebra of V .

Some properties of the tensor algebra $T(V)$:

- * associative
- * graded as $T^m V \times T^n V \rightarrow T^{m+n} V$
- * infinite-dim
- * generated as an algebra by any basis of V

The tensor algebra of V is characterized by this universal property: for any associative unital algebra A and any linear map $\phi: V \rightarrow A$, there is a unique algebra morphism $\psi: T(V) \rightarrow A$ such that the diagram



commutes.

Let I be the two-sided ideal in $T(V)$ generated by the set $\{x \otimes y - y \otimes x \mid x, y \in V\}$

generated
the intersection of all two-sided ideals containing these elements

The symmetric algebra of V is the quotient $S(V) \stackrel{\text{def}}{=} T(V)/I$

This is a commutative algebra with the same universal property as $T(V)$ but restricted to commutative algebras.

If x_1, x_2, \dots, x_n is a basis for V then $S(V)$ is isomorphic to the polynomial algebra $\mathbb{F}[x_1, x_2, \dots, x_n]$ in n commuting variables.

The tensor algebra $T(V)$ is similarly isomorphic to the free associative algebra $\mathbb{F}\langle x_1, x_2, \dots, x_n \rangle$ of polynomials in n noncommuting variables.

Proof of existence of universal enveloping algebras (for Lie algebra L)

Let J be the two-sided ideal in $T(L)$ generated by

$$\text{the set } \left\{ \underbrace{x \otimes y}_{\in T^2 L} - \underbrace{y \otimes x}_{\in T^2 L} - \underbrace{[x, y]}_{\in T^1 L} \mid x, y \in L \right\}.$$

Next we set $U(L) \stackrel{\text{def}}{=} T(L) / J$. Also define

$\pi : T(L) \rightarrow U(L)$ to be the quotient map and define

$i : L \rightarrow U(L)$ to be the composition $L \hookrightarrow T(L) \xrightarrow{\pi} U(L)$.

Since $J \subseteq \bigoplus_{n=2}^{\infty} T^n L$ the quotient $U(L)$ is nonzero and contains $T^0 L = \mathbb{F}$

It is not yet clear whether or not i is injective

(this will turn out to be true but is not part of any defs)

To show that $(U(L), i)$ is a universal enveloping algebra:

suppose (A, j) is some enveloping algebra for L .

The universal property of $T(L)$ gives us a unique algebra

homomorphism $\phi': T(L) \rightarrow A$ such that the diagram

$$\begin{array}{ccc} L & \hookrightarrow & T(L) \\ & j \searrow & \downarrow \phi' \\ & & A \end{array} \quad \text{Commutates.}$$

But all elements $x \otimes y - y \otimes x - [x, y]$ for $x, y \in L$ are in $\ker(\phi')$,

since $\phi'(x \otimes y) = \phi'(x)\phi'(y)$ as ϕ' is algebra hom. Thus $J \subseteq \ker(\phi')$

so ϕ' descends to the desired unique morphism $(U(L), i) \rightarrow (A, \phi)$. \square

Ex. Suppose L is abelian so that $[x, y] = 0 \forall x, y \in L$.

Then $J = I$ and $U(L) = S(L)$ is the

Symmetric algebra of L .

Next: algebra structure of $U(L)$



and the Poincaré-Birkhoff-Witt theorem

describing a basis for $U(L)$