MATH 5143 - Lecture 16

Last time: we discussed enveloping algebras, universal envel. alg., tensor algebras, symmetric algebras. For most of this discussion, $\mathbb{F}$ is any field, L is any Lie algebra over $\mathbb{F}$, V is any $\mathbb{F}$-vector space.

Recall motivation: the universal envel. alg. $\mathcal{U}(L)$ of L will give an associative unital algebra whose repns are "the same" as L-repns, and from which we can construct all L-repns

Def An <u>enveloping algebra</u> (for L) is a pair (A, ϕ) where A is an (associative unital) algebra, $\phi : L \to A$ is a linear map with $\phi([x,y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$ $\forall x,y \in L$

A <u>morphism</u> $(A, \phi) \xrightarrow{f} (B, \psi)$ is an alg. morphism $f : A \to B$ such that

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ & L \arrow[ul, "\phi"] \arrow[u, "\psi"'] \end{tikzcd}$$

commutes.

Def. A <u>universal enveloping algebra</u> (for L) is an $\boxed{\text{initial}}$ object in the category of envel. algebras.

an object with a unique morphism to any other object.

How to construct such an initial object (from last time):

Let $T(L) = \underbrace{\mathbb{F}}_{T^0(L)} \oplus \underbrace{L}_{T^1(L)} \oplus \underbrace{(L \otimes L)}_{T^2(L)} \oplus \underbrace{(L \otimes L \otimes L)}_{T^3(L)} \oplus \cdots$
be the tensor algebra of L (with product given by $\otimes$)

Form J as the two-sided ideal of $T(L)$ generated by the set $\{ x \otimes y - y \otimes x - [x,y] \mid x, y \in L \}$

intersection of all ideals containing the set

Now let $\mathcal{U}(L) \stackrel{\text{def}}{=} T(L)/J$ and let $i : L \to \mathcal{U}(L)$
be the linear map formed by composing

$$L = T^1(L) \hookrightarrow T(L) \quad \text{and} \quad T(L) \xrightarrow{\pi} \mathcal{U}(L)$$

Main thm from last time $(\mathcal{U}(L), i)$ is a universal enveloping algebra for L, and every other univ. envelop. alg. for L is uniquely isomorphic to $(\mathcal{U}(L), i)$

Ex If L is abelian then $J = \langle x \otimes y - y \otimes x \mid x, y \in L \rangle$ and $\mathcal{U}(L) = S(L) =$ the symmetric algebra on L

Goals for today: understand the structure of $\mathcal{U}(L)$, show e.g. that $i : L \to \mathcal{U}(L)$ is injective.

$$\mathcal{U}(L) \overset{\text{def}}{=} T(L) \,/\, \underbrace{\langle x \otimes y - y \otimes x - [x,y] \mid x,y \in L \rangle}_{=J}$$

Some related notation:

Let $T_m \overset{\text{def}}{=} \bigoplus_{k=0}^{m} T^k(L) = T^0(L) \oplus T^1(L) \oplus \cdots \oplus T^m(L)$ and

$\mathcal{U}_{-1} \overset{\text{def}}{=} 0$ and $\mathcal{U}_m \overset{\text{def}}{=} \pi(T_m)$ where $\pi : T(L) \xrightarrow{\text{quotient}} \mathcal{U}(L)$

Clearly $\mathcal{U}_m \cdot \mathcal{U}_n \subseteq \mathcal{U}_{m+n}$ and $\mathcal{U}_m \subseteq \mathcal{U}_{m+1}$

so we can define a vector space $G^m \overset{\text{def}}{=} \mathcal{U}_m / \mathcal{U}_{m-1}$ and

set $G(L) \overset{\text{def}}{=} \bigoplus_{m \geq 0} G^m \neq \mathcal{U}(L)$. There is a well-defined associative bilinear map $G^m \times G^n \to G^{m+n}$ so

we can view $G(L)$ as a graded associative algebra.

There is also a surjective linear map $T(L) \to G(L) = \bigoplus_{m \geq 0} U_m / U_{m-1}$ given by summing all of the maps

$$\phi_m : T^m(L) \xrightarrow{\pi} U_m \xrightarrow{\text{quotient}} G^m = U_m / U_{m-1}$$

This map is surjective because $\pi(T_m \setminus T_{m-1}) = U_m \setminus U_{m-1}$

<u>Lemma</u> The map $\phi = \bigoplus_{m \geq 0} \phi_m$ is an algebra morphism $T(L) \to G(L)$ with $\phi(I) = 0$ where $I = \langle x \otimes y - y \otimes x \mid x, y \in L \rangle$ so ϕ descends to an algebra morphism $S(L) = T(L)/I \to G(L)$

Pf Let $x = x_1 \otimes \cdots \otimes x_m \in T^m(L)$ and $y = y_1 \otimes \cdots \otimes y_n \in T^n(L)$

Then $\phi(xy) = \phi(x)\phi(y)$ so ϕ is an algebra morphism.

$$\begin{array}{ccc} \phi(xy) & = & \phi(x)\phi(y) \\ \| & & \| \\ \phi_{m+n}(xy) & = & \phi_m(x)\phi_n(y) \end{array}$$

For any $x, y \in L$ we have $\pi(x \otimes y - y \otimes x) \in \mathcal{U}_2$ but $\pi(x \otimes y - y \otimes x) = \pi([x, y]) \in \mathcal{U}_1$ so it follows that

$$\phi(x \otimes y - y \otimes x) \in \mathcal{U}_1 / \mathcal{U}_1 = 0 \quad \text{so} \quad I \subseteq \ker \phi. \ \square$$

This lemma leads to the following fundamental result:

Thm [PBW thm] The algebra morphism $\omega : S(L) \to G(L)$ induced by ϕ is an isomorphism.

Detailed proof is in the textbook ... (skip in lecture)

We are more interested in the consequences of the PBW thm.

Cor. Let W be a subspace of $T^m(L)$. Suppose the quotient map $T^m(L) \to S^m(L)$ sends W isomorphically onto $S^m(L)$. Then $\mathcal{U}_m = \mathcal{U}_{m-1} \oplus \pi(W)$.

[What is $S^m(L)$? This just the image of $T^m(L)$ under the quotient map $T(L) \to S(L)$. We have $S(L) = \bigoplus_{m \geq 0} S^m(L)$]

Pf. Consider the diagram

$$\begin{tikzcd} T^m(L) \arrow[r, "\|"] \arrow[dr, "\text{quotient}"'] & U_m \arrow[r, "\text{quotient}"] & G^m = U_m/U_{m-1} \\ & S^m(L) \arrow[ur, "w"'] & \end{tikzcd}$$

The lemma and PBW thm imply that this diagram commutes, so as w is an isomorphism, the bottom two maps must send W isomorphically onto G^m. [Note: U_{m-1} is kernel of $U_m \to G^m$.] □

<u>Cor</u> The map $i: L \hookrightarrow T(L) \xrightarrow{\pi} \mathcal{U}(L)$ is injective.

<u>Pf</u> If we take $W = T^1(L) = L$ then the quotient map $T(L) \to S(L)$ sends W isomorphically onto $S^1(L) = T^1(L)$ so prev. corollary implies that $\pi(L) \oplus \mathcal{U}_0 = i(L) \oplus \mathbb{F} = \mathcal{U}_1$ so $i(L)$ is complementary to $\mathcal{U}_0$ in $\mathcal{U}_1$ and i is injective $\square$

<u>Cor</u> If $(\mathcal{U}, i)$ is any universal enveloping algebra for L then i is injective.

<u>Pf</u> (Because all univ. env. alg. are $\cong$) $\square$

Cor Suppose $x_1, x_2, x_3, \ldots$ is an ordered basis for L. Then a basis for $\mathcal{U}(L)$ is provided by all elements

(*) $x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_m} \overset{\text{def}}{=} \pi(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_m})$

where $m \geq 0$ and $i_1 \leq i_2 \leq \cdots \leq i_m$.

[In this setting the case $m=0$ contributes the unit 1.]

Call the set of elems (*) the $\boxed{\text{PBW basis}}$ of $\mathcal{U}(L)$

Pf Let W be the subspace of $T^m(L)$ spanned by the PBW basis elements of degree m. Then W is mapped isomorphically onto $S^m(L)$ and so the corollary above implies that $\pi(W)$ is complementary to $\mathcal{U}_{m-1}$ in $\mathcal{U}_m$. By induction on m, it follows that the PBW basis spans $\mathcal{U}(L)$ and is linearly independent $\square$

Ex. $x_1 \cdot x_2 = x_1 x_2$ but

$$x_2 \cdot x_1 = x_1 \cdot x_2 + \underbrace{[x_2, x_1]}_{\in L = \mathbb{F}\text{-span}\{x_1, x_2, \dots\}} = x_1 x_2 + \sum_i a_i x_i \text{ for some } a_i \in \mathbb{F}.$$

Cor Suppose H is a subalgebra of L with an ordered basis $(h_1, h_2, \dots)$ that can be extended to a basis of L by adding $(x_1, x_2, \dots)$. Then the inclusion $H \hookrightarrow L$ extends to an injective algebra morphism $\mathcal{U}(H) \hookrightarrow \mathcal{U}(L)$ and $\mathcal{U}(L)$ is a free $\mathcal{U}(H)$-module with basis given by the PBW basis elements only involving $x_1, x_2, x_3, \dots$.

Pf Clear from the description of the PBW basis $\square$

Free Lie algebras $\longrightarrow$ analogous to free groups

Suppose our Lie algebra L is generated by a set X, meaning $X \subset L$ and there is no proper Lie subalgebra containing X.

<u>Def</u> L is <u>free</u> on X if for any map $\phi : X \to M$ where M is a Lie algebra, there exists a unique Lie alg. morphism $\psi : L \to M$ such that

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\text{inclusion}"'] & M \\ & L \arrow[u, "\psi"'] \end{tikzcd}$$

commutes.

Usual universal property arguments show that any two Lie algebras that are free on isomorphic sets are isomorphic

(same size sets)

Given a set X, how to form a free Lie alg. on X?

① Let V be a vector space with X as basis.

② Form tensor algebra $T(V)$ viewed as a Lie algebra with $[a,b] = ab - ba$.

③ Let L be the Lie subalgebra of $T(V)$ generated by X.

Claim: This gives a free Lie algebra L on X.

Pf Suppose $\phi : X \to M$ is a map with M a Lie algebra. First extend ϕ to a linear map $V \to M \subset \mathcal{U}(M)$. Then canonically extend this to algebra morphism $T(V) \to \mathcal{U}(M)$, and restrict this to a Lie algebra morphism. [Some details left to check] □

Def If L is free on X and R is the ideal of L generated by some elements $\{f_j \mid j \in I\}$ then we call $L/R = \langle X \mid f_j = 0 \ \forall j \in I \rangle$ the Lie algebra generated by X with relations $f_j = 0$.