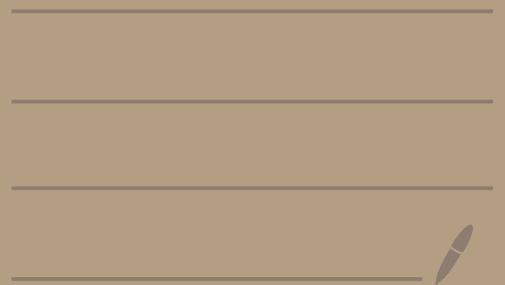


MATH 5143 - Lecture 16



Last time: we discussed enveloping algebras, universal envel. alg., tensor algebras, symmetric algebras. For most of this discussion, \mathbb{F} is any field, L is any Lie algebra over \mathbb{F} , V is any \mathbb{F} -vector space.

Recall motivation: the universal envel. alg. $U(L)$ of L will give an associative unital algebra whose reps are "the same" as L -reps, and from which we can construct all L -reps

Def An enveloping algebra (for L) is a pair (A, ϕ) where A is an (associative unital) algebra, $\phi: L \rightarrow A$ is a linear map with $\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$ $\forall x, y \in L$

A morphism $(A, \phi) \xrightarrow{f} (B, \psi)$ is an alg. morphism

$f: A \rightarrow B$ such that $A \xrightarrow{f} B$ commutes.

$$\begin{array}{ccc} & & \\ & \phi \uparrow & \\ & & L \xrightarrow{\psi} \\ & & \end{array}$$

Def. A universal enveloping algebra (for L) is an

initial object in the category of envel. algebras.
 an object with a unique morphism to any other object.

How to construct such an initial object (from last time):

Let $T(L) = \mathbb{F} \oplus L \oplus (L \otimes L) \oplus (L \otimes L \otimes L) \oplus \dots$
be the tensor algebra of L (with product given by \otimes)

Form J as the two-sided ideal of $T(L)$
generated by the set $\{x \otimes y - y \otimes x - [x, y] \mid x, y \in L\}$
intersection of all ideals containing the set

Now let $U(L) \stackrel{\text{def}}{=} T(L) / J$ and let $i: L \rightarrow U(L)$
be the linear map formed by composing

$$L = T^1(L) \hookrightarrow T(L) \quad \text{and} \quad T(L) \xrightarrow{\pi} U(L)$$

Main thm from last time $(U(L), i)$ is a universal enveloping algebra for L , and every other univ. envelop. alg. for L is uniquely isomorphic to $(U(L), i)$

Ex If L is abelian then $J = \langle x \otimes y - y \otimes x \mid x, y \in L \rangle$ and $U(L) = S(L) =$ the symmetric algebra on L

Goals for today: understand the structure of $U(L)$, show e.g. that $i: L \rightarrow U(L)$ is injective.

$$U(L) \stackrel{\text{def}}{=} T(L) / \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in L \rangle$$

Some related notation: $= J$

$$\text{Let } T_m \stackrel{\text{def}}{=} \bigoplus_{k=0}^m T^k(L) = T^0(L) \oplus T^1(L) \oplus \dots \oplus T^m(L) \text{ and}$$

$$U_{-1} \stackrel{\text{def}}{=} 0 \text{ and } U_m \stackrel{\text{def}}{=} \pi(T_m) \text{ where } \pi: T(L) \xrightarrow{\text{quotient}} U(L)$$

Clearly, $U_m \cdot U_n \subseteq U_{m+n}$ and $U_m \subseteq U_{m+1}$

so we can define a vector space $G^m \stackrel{\text{def}}{=} U_m / U_{m-1}$ and

set $G(L) \stackrel{\text{def}}{=} \bigoplus_{m \geq 0} G^m \neq U(L)$. There is a well-defined associative bilinear map $G^m \times G^n \rightarrow G^{m+n}$ so

we can view $G(L)$ as a graded associative algebra.

There is also a surjective linear map $T(L) \rightarrow G(L) = \bigoplus_{m \geq 0} U_m / U_{m-1}$
 given by summing all of the maps

$$\phi_m : T^m(L) \xrightarrow{\pi} U_m \xrightarrow{\text{quotient}} G^m = U_m / U_{m-1}$$

This map is surjective because $\pi(T_m \setminus T_{m-1}) = U_m \setminus U_{m-1}$

Lemma The map $\phi = \bigoplus_{m \geq 0} \phi_m$ is an algebra morphism

$T(L) \rightarrow G(L)$ with $\phi(I) = 0$ where $I = \langle x \otimes y - y \otimes x \mid x, y \in L \rangle$

so ϕ descends to an algebra morphism $S(L) = T(L) / I \rightarrow G(L)$

Pf Let $x = x_1 \otimes \dots \otimes x_m \in T^m(L)$ and $y = y_1 \otimes \dots \otimes y_n \in T^n(L)$

Then $\phi(xy) = \phi(x)\phi(y)$ so ϕ is an algebra morphism.

$$\begin{array}{ccc} \parallel & & \parallel \\ \phi_{m+n}(xy) & = & \phi_m(x)\phi_n(y) \end{array}$$

For any $x, y \in L$ we have $\pi(x \otimes y - y \otimes x) \in \mathcal{U}_2$ but
 $\pi(x \otimes y - y \otimes x) = \pi([x, y]) \in \mathcal{U}_1$ so it follows that

$$\phi(x \otimes y - y \otimes x) \in \mathcal{U}_1 / \mathcal{U}_1 = 0 \quad \text{so } \mathcal{I} \subseteq \ker \phi. \quad \square$$

This lemma leads to the following fundamental result:

Thm [PBW thm] The algebra morphism $\omega: S(L) \rightarrow U(L)$
induced by ϕ is an isomorphism.

Detailed proof is in the textbook... (skip in lecture)

We are more interested in the consequences of the PBW thm.

Cor. Let W be a subspace of $T^m(L)$. Suppose the quotient map $T^m(L) \rightarrow S^m(L)$ sends W isomorphically onto $S^m(L)$. Then $U_m = U_{m-1} \oplus \pi(W)$.

[What is $S^m(L)$? This is just the image of $T^m(L)$ under the quotient map $T(L) \rightarrow S(L)$. We have $S(L) = \bigoplus_{m \geq 0} S^m(L)$]

PF. Consider the diagram

$$\begin{array}{ccccc} T^m(L) & \xrightarrow{\cong} & U_m & \xrightarrow{\text{quotient}} & G^m = U_m/U_{m-1} \\ & & & \nearrow \omega & \\ & & S^m(L) & & \end{array}$$

The lemma and PBW thm

imply that this diagram commutes,

so as ω is an isomorphism, the bottom two maps must send W isomorphically onto G^m . [Note: U_{m-1} is kernel of $U_m \rightarrow G^m$.] \square

Cor The map $i : L \hookrightarrow T(L) \xrightarrow{\pi} U(L)$ is injective.

Pf If we take $W = T'(L) = L$ then the quotient map $T(L) \rightarrow S(L)$ sends W isomorphically onto $S'(L) = T'(L)$ so prev. corollary implies that $\pi(L) \oplus \mathcal{U}_0 = i(L) \oplus \mathbb{F} = \mathcal{U}_1$ so $i(L)$ is complementary to \mathcal{U}_0 in \mathcal{U}_1 and i is injective \square

Cor If (\mathcal{U}, i) is any universal enveloping algebra for L then i is injective.

Pf (Because all univ. env. alg. are \cong) \square

Cor Suppose x_1, x_2, x_3, \dots is an ordered basis for L .

Then a basis for $U(L)$ is provided by all elements

$$(*) \quad x_{i_1} x_{i_2} x_{i_3} \dots x_{i_m} \stackrel{\text{def}}{=} \pi(x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_m})$$

where $m \geq 0$ and $i_1 \leq i_2 \leq \dots \leq i_m$.

[In this setting the case $m=0$ contributes the unit 1.]

Call the set of elements $(*)$ the **PBW basis** of $U(L)$

Pf Let W be the subspace of $T^m(L)$ spanned by the PBW basis elements of degree m . Then W is mapped isomorphically onto $S^m(L)$ and so the corollary above implies that $\pi(W)$ is complementary to U_{m-1} in U_m . By induction on m , it follows that the PBW basis spans $U(L)$ and is linearly independent \square

Ex. $x_1 \cdot x_2 = x_1 x_2$ but

$$x_2 \cdot x_1 = x_1 \cdot x_2 + \underbrace{[x_2, x_1]}_{\in L = \mathbb{F}\text{-span}\{x_1, x_2, \dots\}} = x_1 x_2 + \sum_i a_i x_i \text{ for some } a_i \in \mathbb{F}.$$

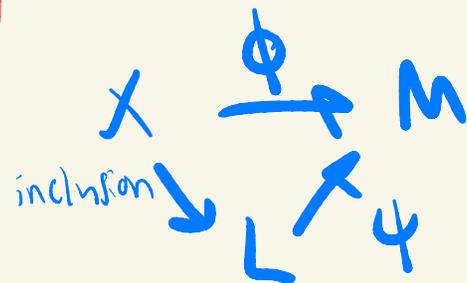
Cor Suppose H is a subalgebra of L with an ordered basis (h_1, h_2, \dots) that can be extended to a basis of L by adding (x_1, x_2, \dots) . Then the inclusion $H \hookrightarrow L$ extends to an injective algebra morphism $\mathcal{U}(H) \hookrightarrow \mathcal{U}(L)$ and $\mathcal{U}(L)$ is a free $\mathcal{U}(H)$ -module with basis given by the PBW basis elements only involving x_1, x_2, x_3, \dots .

Pf Clear from the description of the PBW basis \square

Free Lie algebras \rightarrow analogous to free groups

Suppose our Lie algebra L is generated by a set X , meaning $X \subset L$ and there is no proper Lie subalgebra containing X .

Def L is free on X if for any map $\phi: X \rightarrow M$ where M is a Lie algebra, there exists a unique Lie alg. morphism $\psi: L \rightarrow M$ such that



commutes.

Usual universal property arguments show that any two Lie algebras that are free on isomorphic sets (same size sets) are isomorphic.

Given a set X , how to form a free Lie alg. on X ?

- ① Let V be a vector space with X as basis.
- ② Form tensor algebra $T(V)$ viewed as a Lie algebra with $[a, b] = ab - ba$.
- ③ Let L be the Lie subalgebra of $T(V)$ generated by X .

Claim This gives a free Lie algebra L on X .

Pf Suppose $\phi: X \rightarrow M$ is a map with M a Lie algebra. First extend ϕ to a linear map $V \rightarrow M \subset U(M)$. Then (canonically) extend this to algebra morphism $T(V) \rightarrow U(M)$, and restrict this to a Lie algebra morphism. [Some details left to check] \square

Def If L is free on X and R is the ideal of L generated by some elements $\{f_j \mid j \in I\}$ then we call $L/R = \langle X \mid f_j = 0 \forall j \in I \rangle$ the Lie algebra generated by X with relations $f_j = 0$.