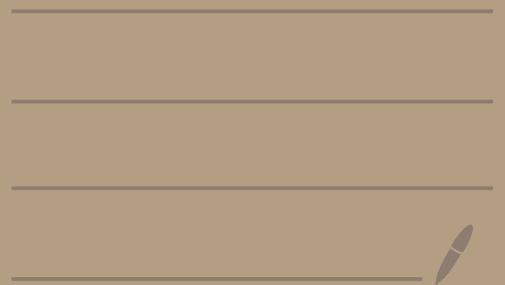# MATH 5143 - Lecture #17

Last time: universal enveloping algebras

For a Lie algebra L (over any field $\mathbb{F}$)

a universal enveloping algebra is an (associative,

unital) algebra $\mathcal{U}(L)$ with a linear map

$i : L \longrightarrow \mathcal{U}(L)$ such that $i([X,Y]) = i(X)i(Y) - i(Y)i(X)$
$$\forall X, Y \in L$$

and with the universal property that if A is any algebra
with a linear map $\phi : L \rightarrow A$ such that $\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$

then $\exists$ unique alg. morphism $\psi : \mathcal{U}(L) \rightarrow A$ with $\phi = \psi \circ i$

Fact Such an object $(U(L), i)$ exists for any L

Fact The universal property determines $U(L)$ up to $\cong$

Fact The map $i : L \to U(L)$ turns out to be always injective

Construction : We can build $U(L)$ by forming the quotient

$T(L) / J$ where $T(L) = \bigoplus_{n \geq 0} T^n L$ is tensor algebra, and

$$J = \left(\text{two-sided ideal generated by all elems } [X,Y] - X \otimes Y + Y \otimes X \atop \text{for } X, Y \in L = T^1 L \right)$$

The map $i : L \to U(L)$ is given by $L \overset{\text{inclusion}}{\hookrightarrow} T(L) \overset{\text{quotient}}{\twoheadrightarrow} U(L)$

Ex If L is abelian, so $[L,L]=0$, then $U(L)=S(L)$, the symmetric algebra on L ($\cong$ polynomials in any fixed basis of L)

Thm (PBW thm) Suppose $x_1, x_2, x_3, \ldots$ is some ordered basis for L. Then a basis for $U(L)$ is provided by the images of the tensors $x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k} \in T(L)$ as $(i_1, i_2, \ldots, i_k)$ varies over all weakly increasing finite integer sequences $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k$. Moreover, if $X \in T^k L$ and $Y \in T^\ell L$ then

$$XY - YX \in \sum_{i=0}^{k+\ell-1} T^i L + J$$

In other words: multiplication in $U(L)$ is commutative mod lower order terms

Formally: the associated graded algebra of $U(L)$ is $\cong S(L)$.

Also from last time: free Lie algebras

The free Lie algebra on a set X is the Lie subalgebra $T(V)$ generated by X, where V is the vector space with X as a basis. One can also specify the free Lie algebra by a universal property.

Write $X = \{x_1, x_2, x_3 \dots\}$. Choose some elements $f_1, f_2, f_3, \dots$ in the free Lie algebra on X. Then we define

$$\langle x_1, x_2, x_3, \dots \mid f_1 = f_2 = f_3 = \dots = 0 \rangle$$

to mean the quotient (free Lie alg. on X) / (two-sided ideal generated by $f_1, f_2, f_3, \dots$)

We call $x_1, x_2, \dots$ the generators and $f_1, f_2, \dots$ the relations of this Lie algebra.

For proof of PBW theorem — see §17.4 of textbook

Jacobi identity plays a role in the argument.

Today: ① sketch statement of Serre's theorem, telling us how to go from abstract root system to semisimple Lie algebra

② Begin studying highest weight repns of semisimple Lie algebras.

Setup everywhere today: L is a semisimple Lie algebra defined over an algebraically closed field of characteristic zero.

L will always be finite-dimensional.

We fix a Cartan subalgebra $H \subseteq L$

same as "maximal toral"
meaning all semisimple elements, abelian, etc.

Choice of H gives us a root space decomposition of L with corresponding root system to be denoted by $\overline{\Phi} \subseteq H^*$

Choose a set of simple roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$
This determines a subset of positive roots $\overline{\Phi}^+ \subsetneq \overline{\Phi}$.

Define $h_j \in H$ such that $\alpha_i(h_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \overset{def}{=} \langle \alpha_i, \alpha_j \rangle$

Finally, choose elements $x_i \in L_{\alpha_i}$ and $y_i \in L_{-\alpha_i}$ with $[x_i, y_i] = h_i$.

Ex. When $L = sl_{n+1}(\mathbb{F})$, we can take H to be the subalgebra of diagonal matrices in L, and then one gets

$$h_i = E_{ii} - E_{i+1,i+1} \quad (1 \leq i \leq n)$$

$$x_i = E_{i,i+1}$$

$$y_i = E_{i+1,i}$$

Prop. In this notation we have $L = \langle x_i, y_i, h_i \mid i = 1,2,\ldots,n \rangle$ and these generators satisfy the following relations:

(S1) $[h_i, h_j] = 0$ ~~plus two more....~~

(S2) $[x_i, y_i] = h_i$ and $[x_i, y_j] = 0$ if $i \neq j$

(S3) $[h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j$ and $[h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j$

$$(S_{ij}^+) \quad (\text{ad} \, x_i)^{1 - \langle \alpha_j, \alpha_i \rangle} (x_j) = 0 \quad \text{for} \quad i \neq j$$

$$(S_{ij}^-) \quad (\text{ad} \, y_i)^{1 - \langle \alpha_j, \alpha_i \rangle} (y_j) = 0 \quad \text{for} \quad i \neq j$$

when $i \neq j$, $\langle \alpha_j, \alpha_i \rangle = \dfrac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\leq 0}$ so there identities make sense

Pf (S1) holds since H is abelian. (S2) by definition and since $[x_i, y_j] \in \underline{L_{\alpha_i - \alpha_j}}$ but $\alpha_i - \alpha_j \notin \underline{\Phi}$ when $i \neq j$. (S3) holds by def.
$$\underbrace{L_{\alpha_i - \alpha_j}}_{= 0}$$

as $[h_i, x_j] = \alpha_j(h_i) x_j$ and $[h_i, y_j] = -\alpha_j(h_i) y_j$.

So the only nontrivial part is checking $(S_{ij}^{+/-})$.

Suffices to check (S_{ij}^+) by symmetry

We assume $i \neq j$. Then $\alpha_j - \alpha_i$ is not a root, so the α_i-root string through α_j the form

$$\alpha_j, \quad \alpha_j + \alpha_i, \quad \alpha_j + 2\alpha_i, \quad \ldots, \quad \alpha_j + q\alpha_i$$

where $q = -\langle \alpha_j, \alpha_i \rangle$ by earlier results.

Why does this imply that $(ad \, x_i)^{1 - \langle \alpha_j, \alpha_i \rangle}(x_j) = 0$?

Just note that $(ad \, x_i)^k$ maps $L_{\alpha_j} \to L_{\alpha_j + k\alpha_i}$

So if $k > -\langle \alpha_j, \alpha_i \rangle$ then $(ad \, x_i)^k(L_{\alpha_j}) = 0$. $\square$

Rmk When rank $\Phi = 1$ so that $L \cong \mathfrak{sl}_2(\mathbb{F})$, then the relations $(S_{ij}^{+/-})$ are vacuous since we have too few indices to have $i \neq j$, and relations (S1)(S2)(S3) reduce to the usual relations

$$[x,y] = h \text{ and } [h,x] = -2x \text{ and } [h,y] = 2y$$

Key point about these relations: they only involve constants depending on the root system Φ.

Thm (Serre's thm) Suppose Φ is any root system with simple system $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Let L be the Lie algebra generated by the $3n$ elements

$$\{x_i, y_i, h_i \mid i = 1, 2, \dots, n\}$$ subject to just the relations

$(S1) (S2) (S3) (S^+_{ij}) (S^-_{ij})$ from previous prop. Then L is a finite-dim semisimple Lie algebra with Cartan subalgebra $H \overset{\text{def}}{=} \mathbb{F}\text{-span}\{h_1, h_2, \dots, h_n\}$ and with corresponding root system Φ (viewing $\Phi \subseteq H^*$ by setting $\alpha_i(h_j) = \langle \alpha_i, \alpha_j \rangle$ and extending by linearity)

We will skip proof out of time constraints.

Rmk If you do this construction but leave out $(s_{ij}^{\pm})$ and (S_{ij}) then the resulting Lie algebra is usually infinite-dimensional, but proof of Serre's thm proceeds by first studying this object.

Cor For each root system $\underline{\Phi}$ there is a semisimple Lie algebra with $\underline{\Phi}$ as its root system (relative to some Cartan subalgebra). Moreover, if we have isomorphic root systems $\underline{\Phi} \xrightarrow{\sim} \underline{\Phi}'$ and with a simple system $\Delta \subseteq \underline{\Phi}$ whose image in $\underline{\Phi}'$ is Δ', then the obvious bijection between generating sets of the associated Lie algebras extends to an isomorphism of Lie algebras.

Pf The described map on generators extends to a Lie algebra homomorphism because the images of the generators of one Lie algebra satisfy the same defining relations in the other Lie algebra. [This a general phenomena of morphisms from Lie algebras constructed by generators and relations: a map from the generating set extends uniquely to a morphism if and only the images of the generators still satisfy all defining relations] You can build such a morphism in either direction, and these must be inverses because their composition is the identity map on the generating sets □

Criteria for semisimplicity

Recall (from HW) that L is <u>reductive</u> if $\text{Rad}(L) = Z(L)$

① Semisimple $\Rightarrow$ reductive since

$Z(L) \subseteq \text{Rad}(L)$ so if $\text{Rad}(L) = 0$ then $Z(L) = 0 = \text{Rad}(L)$

② abelian $\Rightarrow$ reductive since

if $Z(L) = L$ then $\text{Rad}(L) = L = Z(L)$.

Prop If L is reductive then $[L,L]$ is semisimple and $L = [L,L] \oplus Z(L)$

Pf (HW exercise) since $L/Z(L) \cong \text{ad} L$ is semisimple, it acts completely reducibly on L, so $L = M \oplus Z(L)$ for some ideal M. But then $[L,L] = [M,M] \subseteq M$ and we must have equality since $[L/Z(L), L/Z(L)] = L/Z(L)$ as this is semisimple. $\square$

Prop Let $L \subseteq g\ell(V)$, where V is finite-dimensional, be a nonzero Lie algebra acting irreducibly on V. Then L is reductive with $\dim Z(L) \leq 1$ and if $L \subseteq s\ell(V) \subsetneq g\ell(V)$ then L is semisimple.

Pf Let $S = \text{Rad}(L)$. By Lie's thm, S has a common eigenvector in V, call this $v \in V$, with $s \cdot v = \lambda(s) v \ \forall s \in S$ where $\lambda \in S^*$. If $x \in L$ then $[s, x] \in S$ so

must be zero (see next page)

$(*)$ $s \cdot (x \cdot v) = x \cdot (s \cdot v) + [s, x] \cdot v = \lambda(s) x \cdot v + \lambda([s, x]) v$

Since L acts irreducibly on V, all vectors in V are obtained by applying sequences of elements $x \in L$ to $v \in V$ and taking linear combinations. Thus $(*)$ implies that in some basis of V, each $s \in S$ acts as a triangular matrix with $\lambda(s)$ on diagonal.

Thus the trace of any $s \in S$ is $(\dim V)\lambda(s)$.

Because the commutators $[S, L]$ all have trace zero, and because $[S, L] \subseteq S$, λ must vanish on $[S, L]$.

But this means, by looking back at $(*)$, that

$s \in S$ acts exclusively as the scalar $\lambda(s)$

Therefore $\mathrm{Rad}(L) = S \subseteq Z(L)$ whence $\mathrm{Rad}(L) = Z(L)$

so L is reductive and $\dim S \leq 1$.

Finally if $L \subseteq \mathfrak{sl}(V)$ then $S = 0$ since $\mathfrak{sl}(V)$ has no

scalars except zero, in which case L is semisimple. $\square$

Recall the classical Lie algebras:

Type A_n: $\mathfrak{sl}_{n+1}(\mathbb{F}) = (\text{traceless matrices})$

Type C_n: $\mathfrak{sp}_{2n}(\mathbb{F}) \cong \{x \in \mathfrak{gl}_{2n}(\mathbb{F}) \mid Jx + x^T J = 0\}$

$\qquad\qquad\qquad\qquad$ for $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

Types B, D: $\mathfrak{o}_n(\mathbb{F}) \cong \{x \in \mathfrak{gl}_n(\mathbb{F}) \mid x + x^T = 0\}$

Prop Each classical Lie algebra is semisimple and in fact simple.

__Pf__ $\mathfrak{gl}(V) = \mathfrak{sl}(V) + (\text{scalar matrices})$ and $\mathfrak{gl}(V)$

acts irreducibly on V, so $\mathfrak{sl}(V)$ also acts irreducibly.

Thus $\mathfrak{sl}_n(F) \cong \mathfrak{sl}(V)$ for $V = F^n$ is semisimple. $\binom{by}{prop}$

We observed long ago that $\mathfrak{sp}_{2n}(F) \subseteq \mathfrak{sl}_{2n}(F)$ and $\mathfrak{o}_n(F) \subseteq$

$\mathfrak{sl}_n(F)$ so just need to show that these Lie algebras act

irreducibly on F^{2n} or F^n. Straightforward from explicit

Constructions: just want find a way to express any $X \in \mathfrak{gl}_n(F)$

as linear comb. of $\overset{\text{products of}}{\wedge}$ elements in your classical algebra + scalars.

In this way, deduce that each algebra is semisimple.
Simplicity follows by computing the root systems and their Dynkin diagrams $\square$