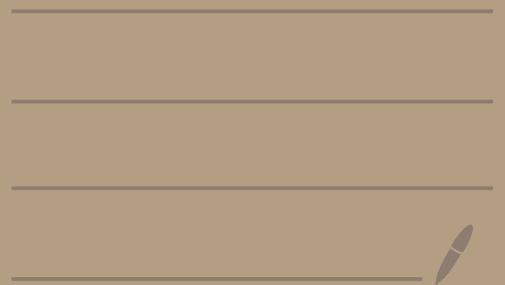


MATH 5143 - Lecture # 18



Representation theory of semisimple Lie algebras

Every where: L is semisimple Lie algebra over field \mathbb{F}
 \mathbb{F} is alg. closed and char. zero.

$\mathfrak{h} \subseteq L$ is a fixed Cartan subalgebra

$\Phi \subseteq \mathfrak{h}^*$ is the corresponding root system

$\Delta \subseteq \Phi$ is a simple system with elems $\alpha_1, \alpha_2, \dots, \alpha_n$

$W \stackrel{\text{def}}{=} \langle r_\alpha \mid \alpha \in \Phi \rangle$ is Weyl group of Φ .

Goal: understand the finite-dimensional L -modules
in particular those which are irreducible.

Suppose V is a finite-dim L module. Then \mathfrak{H} acts on V as commuting diagonalizable operators, so V can be decomposed into simultaneous eigenspaces for \mathfrak{H} .

Specifically, we can write $V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda$

where $V_\lambda \stackrel{\text{def}}{=} \{v \in V \mid h \cdot v = \lambda(h)v \ \forall h \in \mathfrak{H}\}$

If $V_\lambda \neq 0$ (which can only happen for finitely-many $\lambda \in \mathfrak{H}^*$)

then we call V_λ a weight space and λ a weight

Ex If $V = L$, L acting by adjoint repn, then weight spaces are just the root spaces L_α and the weights are the roots $\alpha \in \Phi$

↳ along with \mathfrak{H}

↳ along with 0

Ex If $L = \mathfrak{sl}_2(\mathbb{F}) = \langle x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rangle$

and V is irreducible, then V looks like

$$V = V_{-m} \oplus V_{-m+2} \oplus \dots \oplus V_{m-2} \oplus V_m$$

for some integer $m \geq 0$. Each V_i is a weight

space for the weight $\lambda: h \mapsto i$. Everything

is easy here because $\mathfrak{H} = \mathbb{F}\text{-span}\{h\}$.

Some pathologies: if $\dim V = \infty$ then the sum of the weight spaces $V_\lambda \subseteq V$ may be a proper subspace, though this sum of subspaces is always direct [HW exercise] However:

Lemma Let V be an arbitrary L -module. Then

(a) L_α maps V_λ into $V_{\lambda+\alpha}$ $\forall \lambda \in \mathfrak{H}^*$ and $\alpha \in \Phi$.

(b) $U \stackrel{\text{def}}{=} \sum_{\lambda \in \mathfrak{H}^*} V_\lambda$ is equal to $\bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda$ and is an L -submodule of V .

(c) If $\dim V < \infty$ then $U = V$.

Pf we will just check (a), as (b)(c) are exercises. Note for $x \in L_\alpha$, $v \in V_\lambda$, $h \in \mathfrak{H}$ that $h \cdot x \cdot v = x \cdot h \cdot v + [h, x] \cdot v = (\lambda(h) + \alpha(h)) x \cdot v$ so L_α sends V_λ to $V_{\lambda+\alpha}$. \square

Standard cyclic modules

A maximal vector of weight $\lambda \in \mathfrak{H}^*$ in an L -module V is a nonzero vector $v^+ \in V$ with
$$\begin{cases} X v^+ = 0 \quad \forall \alpha \in \Delta, X \in L_\alpha \\ h v^+ = \lambda(h) v^+ \quad \forall h \in \mathfrak{H} \end{cases}$$

[This depends implicitly on choice of simple roots Δ .

If $\dim V = \infty$ then it could happen that there are no such vectors.]

But if $\dim V < \infty$ then the Borel subalgebra $\mathfrak{B} = \mathfrak{H} \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$

is solvable and so has a common eigenvector in V (by Lie's thm)

and this eigenvector provides a maximal vector (because it is

killed by all L_α for $\alpha \in \Phi^+$). Idea: first study L -modules generated by a maximal vector.

Note that any L -module structure on V corresponds to a map $L \rightarrow \mathfrak{gl}(V)$ which is an algebra, and so extends uniquely to an associative algebra module structure on V relative to $U(L)$.

If $V = U(L) \cdot v^+$ for a maximal vector v^+ of weight λ , then we say V is standard cyclic of weight λ , and we call v^+ the highest weight vector of V .

Fix $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ with $[x_\alpha, y_\alpha] = h_\alpha$ for each $\alpha \in \Phi^+$.

Write $\lambda > \mu$ for $\lambda, \mu \in H^*$ if $\lambda - \mu$ is a sum of positive roots.

Thm Let V be a standard cyclic L -module with highest weight vector $v^+ \in V_{\lambda}$. Write $\Phi^+ = \{\beta_1, \beta_2, \dots, \beta_m\}$ and $\gamma_i \stackrel{\text{def}}{=} \gamma_{\beta_i}$. Then:

(a) V is spanned by the vectors $\gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k} v^+$ as (i_1, i_2, \dots, i_k) ranges over all weakly increasing sequences $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq m$. Also V is the direct sum of its weight spaces

(b) All weights μ for V have the form

$$\mu = \lambda - \sum_{i=1}^m k_i \alpha_i \quad (\text{where } k_i \in \mathbb{Z}_{\geq 0})$$

and therefore $\mu < \lambda$.

(c) For each $\mu \in H^*$, $\dim V_\mu < \infty$ and $\dim V_\lambda = 1$

(d) Each submodule of V is a direct sum of weight spaces

(e) V is an indecomposable L -module with a unique maximal proper submodule whose quotient is irreducible.

(f) every nonzero homomorphic image of V is also standard cyclic of weight λ .

Pf Let $N^- = \bigoplus_{\alpha \in \hat{\Phi}^-} L_\alpha$ and $B = H \oplus \bigoplus_{\alpha \in \hat{\Phi}^+} L_\alpha$ so $L = N^- \oplus B$.

PBW thm implies that $U(L)v^+ = U(N^-)U(B)v^+ = U(N^-)Fv^+$
since v^+ is common eigenvector for B . Part (a) follows from PBW thm for N^- .

Our lemma above implies that $y_{i_1} y_{i_2} \dots y_{i_k} v^+$ has weight

$$(*) \quad \mu = \lambda - \beta_{i_1} - \beta_{i_2} - \dots - \beta_{i_k}$$

So part (b) also follows. There are only finitely many vectors in (a) that can give rise a given weight μ via $(*)$

So $\dim V_\mu < \infty$, and the only such weight vector of weight

λ is v^+ so $\dim V_\lambda = 1$.

For part (d), let W be a submodule of V and write $w \in W$ as a sum of vectors $v_i \in V_{\mu_i}$ for distinct weights μ_i .

We want to show that each v_i is in W .

Suppose otherwise and choose $w = v_1 + \dots + v_n$ with n minimal where none of v_1, v_2, \dots, v_n are in W . (Then $n > 1$)

Find $h \in H$ with $\mu_1(h) \neq \mu_2(h)$. Then $h \cdot w = \sum_i \mu_i(h) v_i \in W$

so $(h - \mu_1(h))w \in W$ but $(h - \mu_1(h))w$ has the form

$$(\mu_2(h) - \mu_1(h))v_2 + \dots + (\mu_n(h) - \mu_1(h))v_n \neq 0$$

contradicting minimality of n . Hence each $v_i \in W$ and (d) holds.

We conclude from (c) and (d) that each proper submodule of V is in sum of weight spaces other than V_λ , so the sum W of all proper submodules is proper, so the quotient V/W must be irreducible.

This proves (e), and (f) holds by definition. \square

Cor If V is as in thm and V is irreducible then v^+ is the unique maximal weight vector up to rescaling.

Pf If there were another such vector of weight λ' then thm implies that $\lambda < \lambda'$ and $\lambda' < \lambda$ so $\lambda = \lambda'$. \square