MATH 5143 - Lecture # 19

Clarifications about universal enveloping algebras

L Lie algebra

$T(L)$ tensor algebra (assoc.) alg. but noncommutative

$S(L)$ symmetric algebra (assoc.) alg. that is commutative

$\mathcal{U}(L)$ univ. enveloping algebra (assoc.) alg. also NOT commutative

What's true: the associated graded algebra of $\mathcal{U}(L)$ is $\cong S(L)$

in $\mathcal{U}(L)$ we have $\underbrace{XY}_{\deg 2} = \underbrace{YX}_{\deg 2} + \underbrace{[X,Y]}_{\deg 1} \neq YX$

key point: multiplication of homogeneous tensors in $\mathcal{U}(L)$ IGNORING LOWER ORDER TERMS is commutative

Representation theory setup: L is a semisimple fin. dim. Lie algebra / $\mathbb{F}$

$H \subseteq L$ is a Cartan subalgebra, $\Phi \subset H^*$ is the root system,

$\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \Phi$ is a chosen base, $\Phi^+ = \{\text{positive roots}\}$,

and $W = \langle r_\alpha \mid \alpha \in \Phi \rangle = \langle r_\alpha \mid \alpha \in \Delta \rangle \subseteq GL(H^*)$

First key observation: Any finite-dim L-module V decomposes as a direct sum of weight spaces $V = \bigoplus_{\lambda \in H^*} V_\lambda$ where $V_\lambda \overset{\text{def}}{=} \{v \in V \mid h \cdot v = \lambda(h) v \ \forall h \in H\}$

Call $\lambda \in H^*$ a weight of V if $V_\lambda \neq 0$ and call V_λ a weight space.

We make same definitions if $\dim V = \infty$ but in that case the sum of weight spaces $\bigoplus_{\lambda \in H^*} V_\lambda$ [which is always direct] may be $\subsetneq V$.

Idea : to avoid pathologies with arbitrary infinite-dim L-modules we consider standard cyclic modules.

A maximal vector in an arbitrary L-module V is a nonzero element v^+ in some weight space of V with $\boxed{x v^+ = 0 \quad \forall x \in L_\alpha \quad \forall \alpha \in \Delta}$

Lie's theorem (applied to the Borel algebra $B = H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$ acting on V) ensures that V has a maximal vector whenever $\dim V < \infty$.

Def A standard cyclic module for L is an L-module V with a maximal vector v^+ such that $V = \mathfrak{U}(L) \cdot v^+$.

In this situation, the maximal vector v^+ will belong to V_λ for some $\lambda \in H^*$, and we say V has highest weight λ and we call v^+ is a highest weight vector

Thm (Structure thm for standard cyclic modules)

For each $\beta \in \Phi^+$ choose $x_\beta \in L_\beta$, $y_\beta \in L_{-\beta}$ such that $[x_\beta, y_\beta] = h_\beta \in H$

Write $\lambda > \mu$ for $\lambda, \mu \in H^*$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta\}$

Suppose V is standard cyclic L-module with maximal vector $v^+ \in V_\lambda$

(a) If $\Phi^+ = \{\beta_1, \beta_2, \beta_3, \dots\}$ then V is spanned by vectors of the form

$$y_{\beta_{i_1}} y_{\beta_{i_2}} y_{\beta_{i_3}} \cdots y_{\beta_{i_k}} v^+ \quad \text{where } 1 \leq i_1 \leq i_2 \leq i_3 \leq \dots \leq i_k$$

(b) All weights μ for V have $\mu < \lambda$ and $\dim V_\mu < \infty$ and $\dim V_\lambda = 1$.

(c) Any submodule of V is a direct sum of its weight spaces

(d) V is indecomposable with unique maximal proper submodule and unique irreducible quotient

(e) Every homomorphic image of V is standard cyclic of same weight λ

Today: first, two more thms about standard cyclic modules

Thm A If V and W are irreducible standard cyclic L-modules with same highest weight $\lambda \in H^*$ then $V \cong W$

Thm B If $\lambda \in H^*$ then there exists an irreducible standard cyclic L-module $V(\lambda)$ of highest weight λ.

Pf of Thm A. Let $X = V \oplus W \overset{\text{def}}{=} \{v+w \mid v \in V, w \in W\}$. This is an L-module and if $v^+ \in V$ and $w^+ \in W$ are highest weight vectors then $x^+ \overset{\text{def}}{=} v^+ + w^+ \in X$ is a maximal vector also of weight λ.

Let Y be the submodule of X generated by x^+. This is standard cyclic by def. But $V \cong Y / \ker \pi_1$ and $W \cong Y / \ker \pi_2$ where $\pi_1 : Y \to V$ and $\pi_2 : Y \to W$ are the obvious surjective homomorphisms. This means V and W are both isomorphic to the unique irreducible quotient of Y. $\square$

To prove Thm B we must explain how to construct standard cyclic modules

Induced modules Begin with a 1-dim vector space $D_\lambda = \mathbb{F}\text{span}\{v^+\}$ spanned by some vector v^+. Let $\lambda \in H^*$ and $B = B(\Delta) \stackrel{\text{def}}{=} H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha \subset L$.

The Borel subalgebra B acts on D_λ linearly by

$$h \cdot v^+ \stackrel{\text{def}}{=} \lambda(h) v^+ \quad \text{and} \quad x v^+ \stackrel{\text{def}}{=} 0 \quad \text{for } h \in H,\ \alpha \in \Phi^+,\ x \in L_\alpha$$

This makes D_λ into a module for B and for $\mathcal{U}(B)$

Def Let $Z(\lambda) \stackrel{\text{def}}{=} \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda$

This a general construction of a $\mathcal{U}(L)$-module: $\mathcal{U}(L)$ + D_λ

left $\mathcal{U}(L)$-mod, right $\mathcal{U}(B)$-mod; left $\mathcal{U}(B)$-mod

Concretely, $Z(\lambda)$ is vector space spanned by the tensors $x \otimes y$ $(x \in \mathcal{U}(L),\ y \in D_\lambda)$ subject to relations

$$(x+x')\otimes y = x \otimes y + x' \otimes y \qquad c(x\otimes y) = (cx)\otimes y = x \otimes (cy)$$

$$x \otimes (y+y') = x\otimes y + x \otimes y' \qquad xb \otimes y = x \otimes by \quad \text{for } x \in \mathcal{U}(L),\ b \in \mathcal{U}(B),\ y \in D_\lambda$$

The way L acts on $Z(\lambda)$ is $A\cdot(x\otimes y) \overset{\text{def}}{=} (Ax)\otimes y$ (w/ highest weight vector $1\otimes v^+$)

Claim $Z(\lambda)$ is a standard cyclic L-module of weight λ.

Pf Every $y\in D_\lambda$ is a scalar multiple of v^+ so every tensor $x\otimes y\in Z(\lambda)$ is equal to $\tilde{x}\cdot(1\otimes v^+)$ where $\tilde{x}\in\mathfrak{U}(L)$ is a scalar multiple of x. For $x\in L_\alpha$ for $\alpha\in\Delta$ we have $x\in B$ so

$$x\cdot(1\otimes v^+) = x\otimes v^+ = 1\otimes xv^+ = 1\otimes 0 = 0.$$

Also for $h\in H\subset B$ we have

$$h\cdot(1\otimes v^+) = h\otimes v^+ = 1\otimes hv^+ = 1\otimes\lambda(h)v^+ = \lambda(h)(1\otimes v^+) \qquad \square$$

Let $N^- = \bigoplus_{\alpha \in -\Phi^+} L_\alpha$. The relation $xb \otimes v^+ = x \otimes bv^+ \ \forall b \in B$, since $L = N^- \oplus B$,

implies that if $\Phi^+ = \{\beta_1, \beta_2, \beta_3, \dots\}$ and $\left\{\begin{array}{l} x_i = x_{\beta_i} \text{ spans } L_{\beta_i} \\ y_i = y_{\beta_i} \text{ spans } L_{-\beta_i} \end{array}\right\}$ then

$$\left\{ y_{i_1} y_{i_2} \cdots y_{i_k} \otimes v^+ \;\middle|\; k \geq 0 \text{ and } i_1 \leq i_2 \leq \cdots \leq i_k \right\}$$

is a basis for $Z(\lambda)$, via the PBW theorem.

Prop $Z(\lambda) \cong \mathcal{U}(L)/I(\lambda)$ as $\mathcal{U}(L)$-modules, where $I(\lambda)$ is left ideal generated in $\mathcal{U}(L)$ by the elements

$$\{x_1, x_2, x_3, \dots\} \cup \{h_\alpha - \lambda(h_\alpha) \cdot 1 \mid \alpha \in \Phi\}$$

Pf These generators annihilate $1 \otimes v^+$ so there is a surjective morphism $\mathcal{U}(L)/I(\lambda) \to Z(\lambda)$ which is injective using PBW theorem. □

<u>Thm</u> (Thm B) Define $V(\lambda)$ for $\lambda \in H^*$ to be the unique irreducible quotient of the standard cyclic module $Z(\lambda)$.

Then $V(\lambda)$ is standard cyclic of weight λ and irreducible.

Note: $V(\lambda)$ still might be infinite-dimensional

<u>Pf</u> Since $Z(\lambda)$ is standard cyclic, and since $V(\lambda)$ as a quotient is a homomorphic image of $Z(\lambda)$, everything follows from structure theorem for standard cyclic modules. $\square$

In some sense, hardest part of thm is showing $Z(\lambda) \neq 0$
(but we will not discuss this issue in detail, follows from PBW thm)

Two new goals: ① Explain when $V(\lambda)$ is finite dim.

② Determine weight spaces $V(\lambda)_\mu \subseteq V(\lambda)$

<u>Fact</u> If V is any irreducible L-module with $\dim V < \infty$ then $V \cong V(\lambda)$ for some $\lambda \in H^*$.

<u>Pf</u> If $\dim V < \infty$ then Lie's thm applied to B-action on V implies existence of a maximal vector of some weight λ. This vector must generate V by irreducibility, so $V \cong V(\lambda)$ by Thm A. $\square$

To be continued next time.