MATH 5143 - Lecture #20

Representation theory setup: L is a semisimple fin. dim. Lie algebra / $\mathbb{F}$

$H \subseteq L$ is a Cartan subalgebra, $\Phi \subset H^*$ is the root system,

$\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \Phi$ is a chosen base, $\Phi^+ = \{\text{positive roots}\}$,

and $W = \langle r_\alpha \mid \alpha \in \Phi \rangle = \langle r_\alpha \mid \alpha \in \Delta \rangle \subseteq GL(H^*)$

An L-module V is standard cyclic of weight $\lambda \in H^*$ if $\exists\, 0 \neq v^+ \in V$ such that $V = \mathcal{U}(L)v$ and $\begin{cases} xv^+ = 0 & \forall \alpha \in \Phi^+\ \forall x \in L_\alpha \\ hv^+ = \lambda(h)v^+ & \forall h \in H \end{cases}$

Thm A If V and W are irreducible standard cyclic L-modules with same highest weight $\lambda \in H^*$ then $V \cong W$

Thm B If $\lambda \in H^*$ then there exists an irreducible standard cyclic L-module $V(\lambda)$ of highest weight λ.

Fact If V is any irreducible L-module with $\dim V < \infty$ then $V \cong V(\lambda)$ for some $\lambda \in H^*$.

Pf If $\dim V < \infty$ then Lie's thm applied to B-action on V implies existence of a maximal vector of some weight λ.
This vector must generate V by irreducibility, so $V \cong V(\lambda)$ by Thm A. $\square$

Goals for today: ① Explain when $V(\lambda)$ is finite dim.
(and next week): ② Determine weight spaces $V(\lambda)_\mu \subseteq V(\lambda)$

For each simple root $\alpha_i \in \Delta$ let $S_i = S_{\alpha_i} = L_{-\alpha_i} \oplus \mathbb{F} h_{\alpha_i} \oplus L_{\alpha_i} \cong \mathfrak{sl}_2(\mathbb{F})$

Then $V(\lambda)$ is a module for S_i and a maximal vector for L is also maximal for S_i

Thm If $V \cong V(\lambda)$ and $\dim V < \infty$ then $\lambda(h_{\alpha_i}) \in \mathbb{Z}_{\geq 0}$ $\forall \alpha_i \in \Delta$
and if $\mu \in H^*$ is any weight for V then $\mu(h_{\alpha_i}) \in \mathbb{Z}$ $\forall \alpha_i \in \Delta$

Pf sketch Follows from $\mathfrak{sl}_2$-repn theory, as V decomposes as sum of fin. dim irr. S_i-modules.

Call $\lambda \in H^*$ $\begin{cases} \underline{\text{dominant}} \text{ if } \lambda(h_\alpha) \geq 0 \ \forall \alpha \in \Delta \ (\text{equiv. } \forall \alpha \in \Phi^+) \\ \underline{\text{integral}} \text{ if } \lambda(h_\alpha) \in \mathbb{Z} \ \forall \alpha \in \Delta \ (\text{equiv. } \forall \alpha \in \Phi) \end{cases}$

Then $\lambda \in H^*$ is <u>dominant integral</u> if $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$ $\forall \alpha \in \Delta$

Let Λ be abelian group of integral weights and Λ^+ the subset of dominant integral weights. Note that $\Lambda \supset \Phi$.

For an L-module V let $\Pi(V) \subseteq H^*$ be its set of weights and define $\Pi(\lambda) = \Pi(V(\lambda))$. If $\dim V < \infty$ then $\Pi(\lambda) \subset \Lambda$.

<u>Next main thm</u> Suppose $\lambda \in \Lambda^+$. Then $V(\lambda)$ has finite dimension and the Weyl group $W \subseteq GL(H^*)$ permutes $\Pi(\lambda)$ with $\dim V(\lambda)_\mu = \dim V(\lambda)_{\sigma\mu}$ $\forall \sigma \in W$.

<u>Cor</u> The map $\lambda \mapsto V(\lambda)$ is a bijection from Λ^+ to isomorphism classes of irreducible fin. dim. L-modules. <u>Pf</u> Combine main thm with fact and thm on prev slide (along with Thm A) $\square$

Pf sketch of main thm

Some identities in $\mathcal{U}(L)$: writing $\underbrace{X_i = X_{\alpha_i}}_{\in L_{\alpha_i}}, \underbrace{Y_i = Y_{\alpha_i}}_{\in L_{-\alpha_i}}$ and $h_{\alpha_i} = [X_i, Y_i]$ for $\alpha_i \in \Delta$

(a) $[x_j, y_i^{k+1}] = 0$ when $i \neq j$, $k \geq 0$

(b) $[h_j, y_i^{k+1}] = -(k+1)\,\alpha_i(h_j)\, y_i^{k+1}$ $(k \geq 0)$

(c) $[x_i, y_i^{k+1}] = -(k+1)\, y_i^k (k - h_i)$ $(k \geq 0)$

Straightforward algebra by induction on $k \geq 0$.

Now we derive a series of claims.

Claim (1) $y_i^{m_i+1} v^+ = 0$ where $m_i = \lambda(h_i) \in \mathbb{Z}_{\geq 0}$, and $v^+ \in V = V(\lambda)$ is a highest weight vector.

Pf Otherwise can use (a)-(c) to show that $y_i^{m_i+1} v^+$ is a second maximal vector of weight $\neq \lambda$ which is impossible $\square$

Claim (2) V contains a nonzero fin. dim. $S_i = S_{\alpha_i}$-module $\cong \mathfrak{sl}_2(\mathbb{F})$

Pf Consider subspace spanned by $v^+, y_i v^+, y_i^2 v^+, \ldots$

This is finite-dim by claim (1). $\square$

Claim (3) V is a sum of finite-dim S_i-modules

Pf Let V' be the sum of all S_i-submodules of finite dim in V

Then $V' \neq 0$ by claim (2). Check that V' is an L-module, hence $V' = V$

↳ use (a) (b) (c) since V irreducible. $\square$

Claim (4) If $\phi : L \to \mathfrak{gl}(V)$ is repn corresp. to L-module structure on V

then $\phi(x_i)$ and $\phi(y_i)$ are both locally nilpotent (meaning nilpotent

when restricted to a finite-dim subspace)

Pf Each $v \in V$ is in a finite sum of fin. dim. S_i-modules, on which $\phi(x_i), \phi(y_i)$ act as

nilpotent operators, by $\mathfrak{sl}_2$-repn theory. $\square$

Claim (5) Define $\sigma_i \overset{\text{def}}{=} \exp(x_i)\exp(-y_i)\exp(x_i)$.

This is an automorphism of V (as a vector space)

Pf Just need to check that σ_i is well-defined, but this follows from prev claim. $\square$

Claim (6) If μ is a weight of V then $\sigma_i(V_\mu) = V_\nu$

for $\nu \overset{\text{def}}{=} r_{\alpha_i}(\mu)$ with $r_\alpha \in W$ the usual reflection.

Pf Follows from $\mathfrak{sl}_2$-repn theory since V_μ is $\overbrace{\text{fin-dim}}^{\text{by structure thm for standard cyclic modules}}$ S_i-submod, see §7.2 in textbook for explicit argument. $\square$

Claim (7) If $\mu \in \Pi(V) = \Pi(\lambda)$ and $w \in W$ then $w(\mu) \in \Pi(\lambda)$ and $\dim V_{w(\mu)} = \dim V_\mu$

Pf Immediate from Claim (6) as $W = \langle r_{\alpha_i} \mid \alpha_i \in \Delta \rangle$ $\square$

Claim (8) $\Pi(\lambda)$ is finite

Pf $\Pi(\lambda)$ is a subset of the set of W-conjugates of all dominant integral $\mu \in H^*$ with $\mu \leq \lambda$ by claim (7) and structure thm of standard cyclic modules. Results in chapter 13 of textbook imply this set is finite. $\square$

Claim (9) $\dim V < \infty$ since $\Pi(V) = \Pi(\lambda)$ is finite and each $\mu \in \Pi(\lambda)$ has $\dim V_\mu < \infty$ $\square$

$\square$

Multiplicity formula

Fix $\lambda \in \Lambda^+$. Then $V(\lambda)$ is fin. dim. irreducible.

For $\mu \in H^*$ let $m_\lambda(\mu) \overset{\text{def}}{=} \dim V(\lambda)_\mu \in \mathbb{Z}_{\geq 0}$

This is zero if $\mu \notin \Pi(\lambda)$. Call $m_\lambda(\mu)$ the <u>multiplicity</u> of μ in $V(\lambda)$.

If $\mu \in H^*$ and $\mu \notin \Lambda$ then $\mu \notin \Pi(\lambda)$ so $m_\lambda(\mu) = 0$.

<u>Thm</u> (Freudenthal's formula) If $\mu \in \Lambda$ and $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ then

$$\Big((\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta) \Big) m_\lambda(\mu) = 2 \sum_{\alpha \in \Phi^+} \sum_{i=1}^{\infty} m_\lambda(\mu+i\alpha)(\mu+i\alpha, \alpha)$$

and this formula provides an effective algorithm to compute $m_\lambda(\mu)$.

<u>key point</u> (nontrivial, see §22 of textbook): if $\lambda \neq \mu$ then $\|\lambda+\delta\|^2 \neq \|\mu+\delta\|^2$

so can divide both sides by this number

<u>minor point</u> (trivial): $\boxed{m_\lambda(\lambda) = 1}$

Formal characters Want to assign to each fin. dim. L-module a vector (similar to character of a group repn) that identifies its isomorphism class.

Notation let $\mathbb{Z}[\Lambda]$ be the free $\mathbb{Z}$-module with basis given by symbols $\{e^\lambda \mid \lambda \in \Lambda\}$ and make this additive group into a ring by setting $e^\lambda e^\mu = e^{\lambda+\mu}$. Here $\Lambda \subset H^*$ is the infinite set of integral weights, including $0 \in \Lambda$.

Def If $\lambda \in \Lambda^+$ then the formal character of $V \cong V(\lambda)$ is

$$ch_V = ch_\lambda \overset{\text{def}}{=} \sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) e^\mu \in \mathbb{Z}[\Lambda].$$

If V is arb. finite dim. L-module then V has unique decomp. $V \cong V(\lambda_1) \oplus V(\lambda_2) \oplus \dots \oplus V(\lambda_k)$ with each $\lambda_i \in \Lambda^+$ and we set $ch_V = \sum_{i=1}^{k} ch_{\lambda_i}$

Ex If $L = \mathfrak{sl}_2(\mathbb{F})$ then $ch_\lambda = e^\lambda + e^{\lambda-\alpha} + e^{\lambda-2\alpha} + \dots + e^{\lambda - m\alpha}$

where $m = \langle \lambda, \alpha \rangle$ [Here $\alpha = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$, $m = \lambda_1 - \lambda_2$]

Weyl group W acts on $\mathbb{Z}[\Lambda]$ by

$$w \cdot \left(\sum_{\mu \in \Lambda} c_\mu e^\mu \right) = \sum_{\mu \in \Lambda} c_\mu e^{w(\mu)} \quad \text{where } c_\mu \in \mathbb{Z}$$

Cor ch_V is fixed by every $w \in W$. **Pf** $m_\lambda(\mu) = m_\lambda(w(\mu))\ \forall w \in W$.

Prop If $f \in \mathbb{Z}[\Lambda]$ is fixed by all $w \in W$ then f has unique expansion as a finite linear combination of formal characters ch_λ for $\lambda \in \Lambda^+$.

Pf idea: write $f = \sum_{\lambda \in \Lambda} c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{Z}$

all but finitely many c_λ's must be zero. Find a maximal $\lambda \in \Lambda^+$ with $c_\lambda \neq 0$, form $g = f - c_\lambda ch_\lambda$, and argue that you may conclude by induction that g has desired expansion. $\square$

↘ need more to deduce uniqueness (exercise)

Prop Suppose V and W are both finite.dim. L-modules

Then $ch_{V \otimes W} = ch_V ch_W$. [Recall how $V \otimes W$ is an L-module:

$x \cdot (v \otimes w) = xv \otimes w + v \otimes xw$ for $\begin{matrix} v \in V \\ w \in W \\ x \in L \end{matrix}$]

Pf Straightforward exercise. $\square$