MATH 5143 – Lecture #21

$\mathrm{Rad}\,L = 0 \iff$ no solvable ideals $\iff$ no abelian ideals

$\iff L = \bigoplus(\text{simple Lie alg.})$

Setup : L is a semisimple Lie algebra defined over an algebraically closed, char. zero field $\mathbb{F}$ Assume $\dim L < \infty$, let $H \subseteq L$ be a Cartan subalgebra, write $\Phi \subset H^*$ for corresponding root system so that $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$, choose a simple system $\Delta \subseteq \Phi$

If V is any L-module (meaning $[x,y]\cdot v = x\cdot y\cdot v - y\cdot x\cdot v$) then a weight space in V is any nonzero subspace of form $V_\lambda = \{v \in V \mid h\cdot v = \lambda(h)v \ \forall h \in H\}$ for $\lambda \in H^*$ call λ the weight

Last time **Fact** Any finite-dim L-module is direct sum of its weight spaces

A *standard cyclic* L-module [of weight $\lambda \in H^*$] is an L-module V generated by a *maximal vector* v^+ (to be called the highest weight vector) with

$$\begin{cases} x \cdot v^+ = 0 & \forall \alpha \in \Delta \ \forall x \in L_\alpha \\ h \cdot v^+ = \lambda(h) v^+ & \forall h \in H \end{cases}$$

Fact Every finite-dim L-module is standard cyclic
Every standard cyclic L-module is direct sum of its weight spaces

Thm Suppose V is a standard cyclic L-module of weight λ
then all weights μ for V have $\dim V_\mu < \infty$ and $\underbrace{\mu < \lambda}_{\text{means } \lambda - \mu \in \mathbb{Z}_{\geq 0}\text{-span}[\Delta]}$ and $\dim V_\lambda = 1$
also V has a unique maximal proper submodule, and unique irreducible quotient, and all homomorphic images of V are also standard cyclic of weight λ

Thm For each $\lambda \in H^*$ there exists a unique isomorphism class of irreducible standard cyclic L-modules $V(\lambda)$ of weight λ. One explicit construction:

Define $\boxed{Z(\lambda) = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda}$

infinite-dim, not nec. irreducible, but it is standard cyclic of wt λ

where $B = H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$ and $D_\lambda = \mathbb{F}\text{-span}\{v^+\}$

with B acting on D_λ such that $\begin{cases} hv^+ = \lambda(h)v^+ & \forall h \in H \\ xv^+ = 0 & \forall x \in L_\alpha, \alpha \in \Phi^+ \end{cases}$

In this tensor product, we have $xb \otimes v^+ = x \otimes bv^+ \quad \forall b \in \mathcal{U}(B)$

Finally define $V(\lambda) \overset{\text{def}}{=} (\text{unique irreducible quotient of } Z(\lambda))$

Fact If V is any finite dim. irreducible L-module then $V \cong V(\lambda)$ for some $\lambda \in H^*$.

Call $\lambda \in H^*$ dominant if $\lambda(h_\alpha) > 0 \ \forall \alpha \in \Delta$

integral if $\lambda(h_\alpha) \in \mathbb{Z} \ \forall \alpha \in \Delta$

dominant integral if $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \Delta$

Let Λ be set of integral $\lambda \in H^*$, let $\Lambda^+ \subset \Lambda$ be subset of dominant integral $\lambda \in \Lambda$

Here: $h_\alpha \overset{\text{def}}{=} \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ where $\kappa : L \times L \to F$ is killing form
$\kappa(x,y) = \operatorname{trace}(\operatorname{ad} x \operatorname{ad} y)$

and $t_\alpha \in H$ is unique elem with
$\kappa(t_\alpha, h) = \alpha(h) \ \forall h \in H$

Thm Let $\lambda \in H^*$. Then the irreducible L-module $V(\lambda)$ is finite-dimensional if and only if $\lambda \in \Lambda^+$ is dominant integral. Moreover, in this case the Weyl group W of Φ permutes the set of weights in $V(\lambda)$ and weights in same W-orbit all have weight spaces of same dimension.

Define $m_\lambda(\mu) \overset{\text{def}}{=} \dim V(\lambda)_\mu$ for $\mu \in H^*$.

There are various formulas/recurrences to explicitly compute these numbers, e.g. Freudenthal's formula (see Lecture #23)

To make it easier to compute and manipulate the multiplicity function m_λ, we encode it as a formal character:

Let $\mathbb{Z}[\Lambda]$ be $\mathbb{Z}$-span of the symbols e^λ for $\lambda \in \Lambda$ viewed as a ring with $e^\lambda e^\mu \overset{def}{=} e^{\lambda+\mu}$ and $1 = e^0$.

The formal character of any fin. dim. L-module V is

$$ch_V \overset{def}{=} \underbrace{\sum_{\mu \in H^*}}_{\text{finite sum}} m_\lambda(\mu) e^\mu \in \mathbb{Z}[\Lambda]$$

W acts linearly on $\mathbb{Z}[\Lambda]$ by $w \cdot e^\lambda = e^{w\lambda}$ and this action fixes ch_V for any V.

Prop Any $f \in \mathbb{Z}[\Lambda]$ fixed by all $w \in W$ has unique expansion as linear comb. of formal characters $ch_\lambda \overset{def}{=} ch_{V(\lambda)}$

Prop If V and W are two finite-dim L-modules then $ch_{V\otimes W} = ch_V \, ch_W$

Today: Harish-Chandra's theorem

Some technical proofs will just be outlined

Let $\mathcal{Z}$ denote the center of the algebra $\mathcal{U}(L)$:

$$\mathcal{Z} \overset{\text{def}}{=} \{x \in \mathcal{U}(L) \mid xy = yx \ \forall y \in L\}$$

This is a commutative subalgebra, and each of L-module is also a $\mathcal{U}(L)$-module and, by restriction, a $\mathcal{Z}$-module.

Consider the standard cyclic L-module

$$Z(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(B)} D_\lambda$$

for some $\lambda \in H^*$, now viewed as a $\mathfrak{Z}$-module.

If v^+ is a maximal vector in $Z(\lambda)$ and $z \in \mathfrak{Z}$

then
$$\begin{cases} h \cdot z \cdot v^+ = z \cdot h \cdot v^+ = \lambda(h) z \cdot v^+ & \forall h \in H \\ x \cdot z \cdot v^+ = z \cdot x \cdot v^+ = 0 & \forall \begin{smallmatrix} \alpha \in \Delta \\ x \in L_\alpha \end{smallmatrix} \end{cases}$$

Thus $z \cdot v^+$ is also a maximal vector of weight λ.

Therefore $z \cdot v^+$ is a scalar multiple of v^+.

Define $\chi_\lambda : \mathfrak{Z} \to \mathbb{F}$ to be map with $z \cdot v^+ = \chi_\lambda(z) v^+ \ \forall z \in \mathfrak{Z}$

Does not depend on choice of v^+, as all maximal vectors in $Z(\lambda)$ are scalar multiples of each other.

Fact χ_λ is an algebra homomorphism

Pf $\chi_\lambda(z_1 z_2)v^+ = z_1 z_2 \cdot v^+ = z_1 \cdot (z_2 \cdot v^+)$

$= \chi_\lambda(z_2)\, z_1 \cdot v^+ = \chi_\lambda(z_1)\chi(z_2)v^+$. $\square$

Call $\chi_\lambda : \left(Z = \text{center of } \mathfrak{U}(L) \right) \to \mathbb{F}$

the (central) character of $\lambda \in H^*$ [or of $Z(\lambda)$]

These central character χ_λ may coincide for different λ's, and Harish-Chandra's thm will tell us precisely when this happens.

Fact If $z \in \mathfrak{Z}$ and $u \in Z(\lambda)$ is any vector then

$$z \cdot u = \chi_\lambda(z) u.$$

Pf Since v^+ generates $Z(\lambda)$ and z commutes with all elements of L, the result follows. $\square$

Cor The action of $z \in \mathfrak{Z}$ on any submodule of homomorphic image of $Z(\lambda)$ is by the scalar $\chi_\lambda(z)$

Def Two elements $\lambda, \mu \in H^*$ are linked (by $w \in W$)

if $\lambda + \delta = w \cdot (\mu + \delta)$ where $\delta \overset{\text{def}}{=} \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

In this situation we write $\mu \sim \lambda$.

Given $\alpha \in \Phi^+$ choose $0 \neq x_\alpha \in L_\alpha$. Then there exists a unique $y_\alpha \in L_{-\alpha}$ such that if $h_\alpha \overset{\text{def}}{=} [x_\alpha, y_\alpha]$ then $\langle x_\alpha, y_\alpha, h_\alpha \rangle \cong \mathfrak{sl}_2(\mathbb{F})$

via $x_\alpha \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y_\alpha \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$h_\alpha \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Prop Let $\lambda \in \Lambda$, $\alpha \in \Delta$, $m = \langle \lambda, \alpha \rangle \in \mathbb{Z}$.

If $m \geq 0$ then $y_\alpha^{m+1} \otimes_{\mathcal{U}(B)} v^+ \in Z(\lambda)$ is a maximal vector of weight $\lambda - (m+1)\alpha$. [Here, $v^+ \in D_\lambda$ so $1 \otimes_{\mathcal{U}(B)} v^+$ generates $Z(\lambda)$]

Pf Formulas last time tell us that:

For $\alpha \neq \beta$ in Δ: $[x_\beta, y_\alpha^{m+1}] = 0 \Rightarrow x_\beta \cdot y_\alpha^{m+1} \otimes v^+ = y_\alpha^{m+1} \otimes \underbrace{x_\beta v^+}_{=0} = 0$

For any $\alpha, \beta \in \Delta$: $[h_\beta, y_\alpha^{m+1}] = -(m+1)\,\alpha(h_\beta)\, y_\alpha^{m+1}$

$$\Rightarrow h_\beta \cdot y_\alpha^{m+1} \otimes v^+ = -(m+1)\,\alpha(h_\beta)\, y_\alpha^{m+1} \otimes v^+ + y_\alpha^{m+1} \otimes \overbrace{h_\beta v^+}^{=\lambda(h_\beta)}$$

$$= (\lambda - (m+1)\alpha)(h_\beta)\; y_\alpha^{m+1} \otimes v^+ \qquad \square$$

If λ and μ are linked by $r_\alpha \in W$ then $\chi_\lambda = \chi_\mu$

Cor If $\boxed{\lambda \in \Lambda}$, $\alpha \in \Delta$, $\mu = r_\alpha \cdot (\lambda + \delta) - \delta$ where $r_\alpha \in W$

$x \mapsto x - \langle x, \alpha \rangle \alpha$

then $\chi_\lambda = \chi_\mu$ $\left(\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta\right)$

Pf r_α sends $\alpha \mapsto -\alpha$ and permutes $\Phi^+ \setminus \{\alpha\}$, so $r_\alpha \delta - \delta = -\alpha$ and

$\mu = r_\alpha \lambda - \alpha = \lambda - (\langle \lambda, \alpha \rangle + 1)\alpha$. we always have $\langle \lambda, \alpha \rangle \in \mathbb{Z}$.

If $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ then previous prop shows that $Z(\lambda)$ has maximal vector of weight μ.

As $z \in \mathcal{Z}$ acts on this vector by the scalar $\underbrace{\chi_\mu(z)}_{\substack{\text{as the maximal}\\ \text{vector of wt } \mu\\ \text{generates a homomorphic}\\ \text{image of } Z(\mu)}}$ and $\underbrace{\text{also } \chi_\lambda(z)}_{\text{by earlier observations}}$, we must have $\chi_\lambda = \chi_\mu$.

If $\langle \lambda, \alpha \rangle < 0$ then

$$\langle \mu, \alpha \rangle = \langle \lambda, \alpha \rangle - 2(\langle \lambda, \alpha \rangle + 1) = -\langle \lambda, \alpha \rangle - 2$$

is ≥ 0 so we can apply proposition with μ in place of λ to deduce the same conclusion. $\square$

Because $W = \langle r_\alpha \mid \alpha \in \Delta \rangle$ we can conclude:

Cor (Easy direction of Harish-Chandra's thm)

If $\lambda \sim \mu$ where $\lambda \in \Lambda$ then $\chi_\lambda = \chi_\mu$.

Thm (Harish-Chandra's thm) Let $\lambda, \mu \in H^*$.

Then $\chi_\lambda = \chi_\mu$ if and only if $\underbrace{\lambda \sim \mu}_{\substack{\text{means } \lambda+\delta \text{ and } \mu+\delta \\ \text{are in the same } W\text{-orbit}}}$

Outline of proof of Harish-Chandra thm

First part: already know that $\lambda \sim \mu \Rightarrow \chi_\lambda = \chi_\mu$ when $\lambda \in \Lambda$

we want to extend this to a statement allowing any $\lambda \in H^*$

Construct PBW bases of $\mathfrak{U}(L)$ and $\mathfrak{U}(H)$ from the basis $\{h_\alpha \mid \alpha \in \Delta\} \sqcup \{x_\alpha, y_\alpha \mid \alpha \in \Phi^+\}$ for L, under any order putting all y_α's first, then the h_α's, then the x_α's.

Then we can define a linear map $\xi : \mathfrak{U}(L) \to \mathfrak{U}(H)$ sending each PBW basis elem in $\mathfrak{U}(H)$ to itself, every other PBW basis elem to 0.

Since $\prod_{\alpha \in \Phi^+} y_\alpha^{i_\alpha} \prod_{\alpha \in \Delta} h_\alpha^{k_\alpha} \prod_{\alpha \in \Phi^+} x_\alpha^{j_\alpha}$ will either

$$\begin{cases} \text{kill } v^+ \in Z(\lambda) \text{ if any } j_\alpha > 0 \\ \text{send } v^+ \text{ to lower weight space if all } j_\alpha = 0 \text{ and any } i_\alpha > 0 \end{cases}$$

it follows that $\boxed{\chi_\lambda(z) = \lambda(\xi(z))}$ $\forall z \in \mathfrak{Z}$

Now define another Lie algebra homomorphism $\eta : H \to \mathcal{U}(H)$ with $\eta(h_\alpha) = h_\alpha - 1$ $\forall \alpha \in \Delta$. This extends to an algebra automorphism $\eta : \mathcal{U}(H) \to \mathcal{U}(H)$. Define

$$\psi : \mathcal{Z} \xrightarrow{\xi} \mathcal{U}(H) \xrightarrow{\eta} \mathcal{U}(H) \quad (\psi = \eta \circ \xi)$$

We can write $\delta = \sum_{\alpha \in \Delta} \lambda_\alpha$ as sum over fundamental weights λ_α which have $\lambda_\alpha(h_\beta) = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$ for $\alpha, \beta \in \Delta$. Then we have

$$(\lambda+\delta)(h_\alpha - 1) = \underbrace{(\lambda+\delta)(h_\alpha)}_{\lambda(h_\alpha)+1} - \underbrace{(\lambda+\delta)(1)}_{=1} = \lambda(h_\alpha)$$

so $(\lambda+\delta)(\psi(z)) = \lambda(\xi(z))$ $\forall z \in \mathcal{Z}, \lambda \in H^*$

$$\Rightarrow \boxed{(\lambda+\delta)(\psi(z)) = \chi_\lambda(z) \ \forall z \in \mathcal{Z}, \lambda \in H^*}$$

Now check that $\psi(z)$ is W-invariant (using the easy case of theorem and properties of W-orbits in Λ) and use this to conclude that

if $\lambda \sim \mu$ then $(\lambda+\delta)(\psi(z)) = (\mu+\delta)(\psi(z))$

and hence that $\chi_\mu = \chi_\mu$, (for any $\lambda, \mu \in H^*$).

The other half of the theorem remains:

if $\chi_\lambda = \chi_\mu$ then need to show that $\lambda \sim \mu$.

This requires a more involved argument → see §23.3 of textbook. $\square$