MATH 5143 – Lecture #22

Setup : L is a semisimple Lie algebra defined over an algebraically closed, char. zero field $\mathbb{F}$

Assume $\dim L < \infty$, let $H \subseteq L$ be a Cartan subalgebra, write $\Phi \subset H^*$ for corresponding root system so that $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$

choose a simple system $\Delta \subseteq \Phi$

For each $\lambda \in H^*$ define $Z(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(B)} D_\lambda$

where $B = H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$ and $D_\lambda = \mathbb{F}\text{-span}\{v^+\}$

Here B is acting on D_λ such that $\left\{ \begin{array}{ll} h v^+ = \lambda(h) v^+ & (h \in H) \\ X v^+ = 0 & (X \in L_\alpha, \alpha \in \Phi^+) \end{array} \right.$

In this tensor product, we have $xb \otimes v^+ = x \otimes bv^+ \quad \forall\, b \in \mathfrak{U}(B)$

Also define $V(\lambda) \overset{\text{def}}{=} \Big($unique irreducible quotient of $Z(\lambda)\Big)$

Fact If V is any finite dim. irreducible L-module then $V \cong V(\lambda)$ for some $\lambda \in H^*$.

Denote the center of the algebra $\mathfrak{U}(L)$ by

$$\mathfrak{Z} \overset{\text{def}}{=} \{ x \in \mathfrak{U}(L) \mid xy = yx \ \forall y \in L \}$$

This is a commutative subalgebra, and each of L-module is also a $\mathfrak{U}(L)$-module and, by restriction, a $\mathfrak{Z}$-module.

Consider the standard cyclic L-module

$$Z(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(B)} D_\lambda$$

for some $\lambda \in H^*$, now viewed as a $\mathcal{Z}$-module.

If v^+ is a maximal vector in $Z(\lambda)$ and $z \in \mathcal{Z}$

then

$$\begin{cases} h \cdot z \cdot v^+ = z \cdot h \cdot v^+ = \lambda(h)\, z \cdot v^+ & \forall h \in H \\ x \cdot z \cdot v^+ = z \cdot x \cdot v^+ = 0 & \forall \alpha \in \Delta, x \in L_\alpha \end{cases}$$

Thus $z \cdot v^+$ is also a maximal vector of weight λ.

Therefore $z \cdot v^+$ is a scalar multiple of v^+.

Define $\boxed{\chi_\lambda : \mathfrak{Z} \to \mathbb{F}}$ to be map with $\boxed{z \cdot v^+ = \chi_\lambda(z)\, v^+}$ $\forall z \in \mathfrak{Z}$

Does not depend on choice of v^+

Fact χ_λ is an algebra homomorphism

Call $\chi_\lambda : \left(\mathfrak{Z} = \text{center of } \mathfrak{U}(L)\right) \to \mathbb{F}$

the (central) character of $\lambda \in H^*$ [or of $Z(\lambda)$]

Def Two elements $\lambda, \mu \in H^*$ are **linked** (by $w \in W$) if $\lambda + \delta = w \cdot (\mu + \delta)$ where $\delta \overset{\text{def}}{=} \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

In this situation we write $\mu \sim \lambda$.

Thm (Harish-Chandra's thm) Let $\lambda, \mu \in H^*$.
Then $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$

Applications of Harish-Chandra thm

We want to introduce formal characters for $Z(\lambda)$ and similar modules

Let $\mathcal{X}$ be the vector space of all formal $\mathbb{Z}$-linear combinations

$$\sum_{\lambda \in H^*} c_\lambda e^\lambda \qquad (c_\lambda \in \mathbb{Z},\ e^\lambda \text{ is a symbol})$$

which are finitely supported in the sense that there are finitely many $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k \in H^*$ such that

$$c_\lambda \neq 0 \Rightarrow \lambda \leq \lambda_i \text{ for some } i$$

where $\lambda \leq \mu$ means $\mu - \lambda \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta\}$

Then the formal character $\boxed{ch_{Z(\lambda)} \overset{def}{=} \sum_{\mu \in H^*} \dim Z(\lambda)_\mu e^\mu}$ belongs to $\mathcal{X}$.

Fact $\mathcal{X}$ is closed under usual multiplication extending ring structure on $\mathbb{Z}[\Lambda]$. (Since $\lambda_i \le \mu_i \Rightarrow \lambda_1 + \lambda_2 + \dots \le \mu_1 + \mu_2 + \dots$)

We now have a well-defined notion of formal character

$$\mathrm{ch}_V \overset{\text{def}}{=} \sum_{\mu \in H^*} \dim V_\mu \, e^{\mu} \in \mathcal{X}$$

for any standard cyclic L-module V.

Let $p(\lambda)$ for $\lambda \in H^*$ be # of functions $k : \Phi^+ \to \mathbb{Z}_{\geq 0}$ such that $\boxed{\lambda + \sum_{\alpha \in \Phi^+} k(\alpha)\alpha = 0}$

clearly $p(\lambda) = 0$ unless $(-\lambda) \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta\}$

Call p the **Kostant (partition) function** and identify $p \leftrightarrow \sum_{\lambda \in H^*} p(\lambda) e^{\lambda} \in \mathcal{X}$

Also let $q = \prod_{\alpha \in \underbrace{\Phi^+}_{\text{finite}}} \left(e^{\alpha/2} - e^{-\alpha/2}\right)$ call this the Weyl function

symbols since $\pm\alpha/2 \in H^*$

Finally set $f_\alpha = e^0 + e^{-\alpha} + e^{-2\alpha} + \cdots \in \mathcal{X}$ for $\alpha \in \Phi^+$.

$\boxed{\text{Lemma A}}$ (a) $p = \prod_{\alpha \in \Phi^+} f_\alpha$ (b) $\left(e^0 - e^{-\alpha}\right) f_\alpha = e^0$

(c) $q = e^{\delta} \prod_{\alpha \in \Phi^+} \left(e^0 - e^{-\alpha}\right)$ where $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

Pf (a) holds by defn, (b) is basic algebra (c) is clear. □

Lemma B For any $w \in W$ it holds that $wq = \mathrm{sgn}(w)\, q$

acts linearly on $\mathcal{X}$ by $w \cdot e^{\lambda} = e^{w\lambda}$

Pf Suffices to show $r_\alpha q = -q$ for any $\alpha \in \Delta$.

Easy enough:
$$r_\alpha q = \underbrace{r_\alpha\left(e^{\alpha/2} - e^{-\alpha/2}\right)}_{r_\alpha :\ \pm\frac{\alpha}{2} \mapsto \mp\frac{\alpha}{2}}\ \underbrace{r_\alpha\Big(\prod_{\beta \in \Phi^+ \setminus \{\alpha\}} \left(e^{\beta/2} - e^{-\beta/2}\right)\Big)}_{\text{fixed by } r_\alpha}$$
$$= -q. \ \square$$

Lemma C $qpe^{-\delta} = e^{0} = 1$

Pf
$$qpe^{-\delta} = \prod_{\alpha \in \Phi^+}\left(e^{0} - e^{-\alpha}\right) \cdot e^{\delta} \cdot p \cdot e^{-\delta} = \prod_{\alpha \in \phi^+}\left(e^{0} - \alpha^{-\alpha}\right) p$$
$$= \prod_{\alpha \in \phi^+}\left(\left(e^{0} - e^{-\alpha}\right) f_\alpha\right) = \prod_{\alpha \in \phi^+} e^{0} = e^{0} = 1 \quad \text{(using Lemma A)}$$

Lemma D $\mathrm{ch}_{Z(\lambda)} = \sum_{\mu\in H^*} p(\mu-\lambda)e^{\mu} = e^{\lambda}\underset{\sim}{p}$

$$\underset{\sim}{p} = \sum_{\mu\in H^*} p(\mu)e^{\mu}$$

Pf Straightforward from properties of $Z(\lambda)$ □

(see Ex. 20.5 in textbook)

Lemma E $q\,\mathrm{ch}_{Z(\lambda)} = e^{\lambda+\delta}$

Pf $q p e^{-\delta} = e^{0} = 1$ and $\mathrm{ch}_{Z(\lambda)} = e^{\lambda} p$

so $q\,\mathrm{ch}_{Z(\lambda)} = e^{\lambda+\delta} q p e^{-\delta} \underset{\substack{\uparrow \\ \text{Lemma C}}}{=} e^{\lambda+\delta}$ □

Want to express $\mathrm{ch}_\lambda = \mathrm{ch}_{V(\lambda)}$ as linear comb. of $\mathrm{ch}_{Z(\mu)}$'s.

Define M_λ (for $\lambda \in H^*$) to be the family of L-modules V such that (1) V is direct sum of its weight spaces

(2) $\mathfrak{Z}$-action on V is by scalar $\chi_\lambda(z)$

(3) $\mathrm{ch}_V \in \mathcal{X}$

M_λ is closed under taking submodules, homomorphic images, direct sums, contains each standard cyclic module.

Cor (of Harish-Chandra thm) $M_\lambda = M_\mu$ iff $\lambda \sim \mu$

Lemma Suppose $0 \neq V \in M_\lambda$. Then V has a maximal vector.

Pf Since $ch_V \in \mathfrak{X}$, for each weight μ of V, and each $\alpha \in \Phi^+$ there is a maximal $k \in \mathbb{Z}_{\geq 0}$ with $\mu + k\alpha$ still a weight. So we can find a weight μ for V such that $\mu + \alpha$ is not a weight $\forall \alpha \in \Phi^+$, and then any nonzero vector in the corresponding weight space is maximal $\square$

For $\lambda \in H^*$, let $\Theta(\lambda) = \{\mu \in H^* \mid \mu < \lambda \text{ and } \mu \sim \lambda\}$.

Prop Let $\lambda \in H^*$. (a) $Z(\lambda)$ has a composition series.

(b) Each composition factor of $Z(\lambda)$ is $\cong V(\mu)$ for some $\mu \in \Theta(\lambda)$

(c) $V(\lambda)$ occurs as exactly one composition factor.

Pf (a) Nothing to prove if $Z(\lambda)$ is irreducible. (then $Z(\lambda) = V(\lambda)$)

Otherwise $Z(\lambda)$ has a proper nonzero submodule $V \in M_\lambda$.

Since $\dim Z(\lambda)_\lambda = 1$, λ is not a weight of V. So by lemma, V has maximal vector, of some weight $\mu \lneq \lambda$.

V contains homomorphic image W of $Z(\mu)$, so $\chi_\lambda = \chi_\mu \Rightarrow \lambda \sim \mu$

$\Rightarrow \mu \in \Theta(\lambda)$. Continue inductively, repeating same argument applied to W and $Z(\lambda)/W$

(b) Each comp. factor is in M_λ so has a maximal vector and is irreducible, so must be standard cyclic, hence $\cong V(\mu)$ for some $\mu \in \Theta(\lambda)$

(c) Clear since $\dim Z(\lambda)_\lambda = 1$. □

Cor Let $\lambda \in H^*$. Then $ch_{V(\lambda)} = \sum_{\mu \in \Theta(\lambda) = \{\nu \in H^* \mid \nu \leq \lambda,\, \nu \sim \lambda\}} c_\mu \, ch_{Z(\mu)}$ for some coeffs $c_\mu \in \mathbb{Z}$ with $c_\lambda = 1$.

Pf Prop. says we can write $ch_{Z(\lambda)} = ch_{V(\lambda)} + \sum_{\mu \in \Theta(\lambda)} d_\mu \, ch_{V(\mu)}$ where $d_\mu \in \mathbb{Z}_{\geq 0}$. Thus $ch_{V(\lambda)} = ch_{Z(\lambda)} - \sum_{\mu \in \Theta(\lambda)} d_\mu \, ch_{V(\mu)}$ and expanding the RHS recursively gives desired formula. □

Thm (Kostant's formula) Let $\lambda \in \Lambda^+$ then

$$m_\lambda(\mu) = \sum_{\underbrace{w \in W}_{\text{potentially many terms}}} \operatorname{sgn}(w)\, p(\mu+\delta - w(\lambda+\delta))$$

Pf $ch_\lambda = \sum_{\mu \in \Theta(\lambda)} c_\mu \, ch_{Z(\mu)}$ with $c_\lambda = 1$. Lemmas E and B tell us that

$$\begin{cases} q\, ch_\lambda = \sum_{\mu \in \Theta(\lambda)} c_\mu e^{\mu+\delta} \quad \text{and} \\ w(q\, ch_\lambda) = w(q)\, w(ch_\lambda) = \operatorname{sgn}(w)\, q\, ch_\lambda \quad \forall w \in W. \end{cases}$$

But also $w\left(\sum_{\mu \in \Theta(\lambda)} c_\mu e^{\mu+\delta}\right) = \sum_{\mu \in \Theta(\lambda)} c_\mu e^{w(\mu+\delta)}$ since $w \in W$ permutes $\Theta(\lambda)$ while $c_\lambda = 1$, deduce that $c_\mu = \operatorname{sgn}(w)$ if $w^{-1}(\mu+\delta) = \lambda+\delta$

So $$\boxed{q\, ch_\lambda = \sum_{w \in W} \mathrm{sgn}(w)\, e^{w(\lambda+\delta)}}$$

By Lemma C, $ch_\lambda = q\, p\, e^{-\delta} ch_\lambda$

$$= p\, e^{-\delta} \left(\sum_{w \in W} \mathrm{sgn}(w)\, e^{w(\lambda+\delta)} \right)$$

$$= p \sum_{w \in W} \mathrm{sgn}(w)\, e^{w(\lambda+\delta)-\delta}$$

$$= \sum_{w \in W} \mathrm{sgn}(w)\, p\, e^{w(\lambda+\delta)-\delta} \qquad \square$$

Cor $q = \sum_{w \in W} \mathrm{sgn}(w)\, e^{w\delta}$

Pf Take $\lambda = 0$. $\square$

$\boxed{\text{Next time}}$: Weyl character formula