MATH 5143 - Lecture 23

Setup throughout: L is a finite-dim. semisimple Lie alg., defined over alg. closed field $\mathbb{F}$ with char $\mathbb{F}=0$, Choose a Cartan subalgebra $H \subset L$, write $\Phi \subset H^*$ for corresp. root sys., choose a set of simple roots Δ, positive roots Φ^+, write

$$W = \langle r_\alpha \mid \alpha \in \Phi \rangle \subset GL(H^*) \text{ for Weyl group.}$$

Last time: Let $\mathfrak{Z}$ be center of universal enveloping alg. $\mathfrak{U}(L)$

For each $\lambda \in H^*$ we have a standard cyclic L-module $Z(\lambda) \overset{\text{def}}{=} \mathfrak{U}(L) \otimes_{\mathfrak{U}(B)} D_\lambda$

1-dim

Borel subalg.

Fact: There exists a unique algebra hom. $\chi_\lambda : \mathfrak{Z} \to \mathbb{F}$ (called the central character) with $a \cdot u = \chi_\lambda(a) u \ \forall a \in \mathfrak{Z}, u \in Z(\lambda)$

Harish-Chandra's thm gives nec. and suff. condition to have $\chi_\lambda = \chi_\mu$ for $\lambda, \mu \in H^*$. Namely: say that $\lambda, \mu \in H^*$ are linked (and write $\lambda \sim \mu$) if $\lambda+\delta$ and $\mu+\delta$ are in same W-orbit where $\delta = \frac{1}{2}\sum_{\alpha\in\phi^+} \alpha$

Thm For $\lambda, \mu \in H^*$ we have $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$.

Define the formal character of any (standard cyclic) L-module V to be the formal expression $ch_V = \sum_{\mu\in H^*} \dim V_\mu \, e^\mu$.

Here $\begin{cases} V_\mu = \{v \in V \mid h\cdot v = \mu(h)v \ \forall h \in H\} \\ e^\mu \text{ is just a formal symbol} \end{cases}$

The reason for this notation is that we want enable adding and multiplying characters like formal power series (or polynomials) under convention that $e^{\lambda} e^{\mu} = e^{\lambda+\mu}$

For this kind of multiplication to be well-defined the set of nonzero coefficients $c_\mu \neq 0$ in a character $\boxed{\sum_{\mu \in H^*} c_\mu e^\mu}$ must be finitely supported in some sense.

Relevant property: If V is standard cyclic then $ch_V \in \mathcal{X}$ where $\mathcal{X}$ is set of expressions $\sum_{\mu \in H^*} c_\mu e^\mu$ for which there are finitely many $\lambda_1, \lambda_2, \ldots, \lambda_k \in H^*$ such that $c_\mu \neq 0 \Rightarrow \mu < \lambda_i$ for some index i

where $\mu < \lambda$ means $\lambda - \mu \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta\}$.

Recall that $V(\lambda)$ is unique irreducible quotient of $Z(\lambda)$.

$\hookrightarrow$ finite.dim iff $\lambda \in \Lambda^+ =$ (set of dominant integral wts) $\subsetneq H^*$

Now, given $\lambda, \mu \in H^*$ define

$$m_\lambda(\mu) = \dim V(\lambda)_\mu = \left(\begin{array}{l}\text{dim of } \mu \text{ weight space} \\ \text{in } V(\lambda)\end{array}\right)$$

so that

$$ch_{V(\lambda)} \overset{\text{def}}{=} ch_\lambda = \sum_{\mu \in H^*} m_\lambda(\mu) e^\mu$$

Let $\operatorname{sgn}: W \to \{\pm 1\}$ be unique group homomorphism

with $\operatorname{sgn}(r_\alpha) = -1$.

Let $p(\lambda) = \left(\begin{array}{l} \text{\# of ways of writing } -\lambda \text{ as a sum of} \\ \text{positive roots} \end{array} \right)$

$= \left(\begin{array}{l} \text{\# functions } k : \Phi^+ \to \mathbb{Z}_{\geq 0} \\ \text{such that } \lambda + \sum_{\alpha \in \Phi^+} k(\alpha)\alpha = 0 \end{array} \right)$

"Kostant partition function"

(dominant, integral, so $\dim V(\lambda) < \infty$)

Thm (Kostant's formula) If $\lambda \in \Lambda^+$ then

$$m_\lambda(\mu) = \sum_{w \in W} \operatorname{sgn}(w)\, p(\mu + \delta - w(\lambda + \delta))$$

Explicit formula but still less efficient than recursive, less explicit algorithms for computation

In proof of Kostant's formula, we encountered two identities:

Let $q \overset{\text{def}}{=} \prod_{\alpha \in \phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{\delta} \prod_{\alpha \in \phi^+} (1 - e^{-\alpha}) \in \mathcal{X}$.

Then (1) $q \, ch_\lambda = \sum_{w \in W} \operatorname{sgn}(w) \, e^{w(\lambda+\delta)}$

(2) $q = \sum_{w \in W} \operatorname{Sgn}(w) \, e^{w\delta}$ (set $\lambda = 0$ in (1))

Substitung (2) into (1) gives the <u>Weyl Character formula</u>

which is ....

<u>Thm</u> If $\lambda \in \Lambda^+$ then

$$\left(\sum_{w \in W} \operatorname{sgn}(w) e^{w\delta}\right) ch_\lambda = \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\delta)}$$

Thus can compute ch_λ by doing "long division" in ring $\mathcal{X}$ $\rightsquigarrow$ but this is somewhat complicated in practice if $|\Delta|$ is large.

Application: an explicit formula for

$$\deg(\lambda) \overset{\text{def}}{=} \dim V(\lambda) = \sum_{\mu \in H^*} m_\lambda(\mu)$$

only defined for $\lambda \in \Lambda^+$

Let $\mathcal{X}_0 \subset \mathcal{X}$ be $\mathbb{Z}$-span$\{e^\lambda \mid \lambda \in H^*\}$ (so formal chars with finite # of nonzero coeffs c_μ)

Then can define eval : $\mathcal{X}_0 \longrightarrow \mathbb{F}$

$$\sum_\mu c_\mu e^\mu \longmapsto \sum_\mu c_\mu$$

Then $\deg(\lambda) = \mathrm{eval}(ch_\lambda)$.

Also eval : $\mathcal{X}_0 \to \mathbb{F}$ is a ring homomorphism

so $\mathrm{eval}(ch_1 ch_2) = \mathrm{eval}(ch_1)\,\mathrm{eval}(ch_2)$.

For $\alpha \in \Phi$ let $D_\alpha : \mathcal{X}_0 \to \mathcal{X}_0$ be linear map with $\boxed{D_\alpha e^\lambda = (\lambda,\alpha) e^\lambda}$

This is a derivation: $D_\alpha(fg) = D_\alpha(f)g + f D_\alpha(g)$ since

$$D_\alpha(e^\lambda e^\mu) = D_\alpha(e^{\lambda+\mu}) = (\lambda+\mu,\alpha)e^{\lambda+\mu} = D_\alpha(e^\lambda)e^\mu + e^\lambda D_\alpha(e^\mu)$$

Let $D = \prod_{\alpha \in \Phi^+} D_\alpha : \mathcal{X}_0 \to \mathcal{X}_0$ (no longer a derivation)

Let $Q = \sum_{w \in W} \mathrm{sgn}(w)\, e^{w\delta}$

$P = \sum_{w \in W} \mathrm{sgn}(w)\, e^{w(\delta+\lambda)}$

so Weyl char. form. is

$$\underbrace{\boxed{Q \cdot ch_\lambda = P}}$$

We want to apply eval $\circ$ D to both sides of this

Since each D_α is derivation, $Q = e^{-\delta} \prod_{\alpha \in \Phi^+} (e^{\alpha} - 1)$, and

$\mathrm{eval}(e^{\alpha} - 1) = 0$, one can show that this gives

$$\underbrace{\mathrm{eval}(D(Q))\, \overbrace{\mathrm{eval}(ch_\lambda)}^{=\deg(\lambda)}}_{=\, \mathrm{eval}(D(Q \cdot ch_\lambda)) \text{ after cancellations}} = \mathrm{eval}(D(P))$$

This implies that $\deg(\lambda) = \dfrac{\mathrm{eval}(D(P))}{\mathrm{eval}(D(Q))}$

Now observe that $\mathrm{eval}(D(e^{\delta})) = \mathrm{eval}\left(\prod_{\alpha\in\phi^+}(\delta,\alpha)\cdot e^{\delta}\right)$

$$= \prod_{\alpha\in\phi^+}(\delta,\alpha)$$

Similarly $\mathrm{eval}(D(e^{w\delta})) = \prod_{\alpha\in\phi^+}(w\delta,\alpha) = \prod_{\alpha\in\phi^+}(\delta, w^{-1}\alpha)$

But recall that w^{-1} permutes ϕ and sends $\ell(w^{-1}) = \ell(w)$ roots in ϕ^+ to $-\phi^+$ so $\quad = (-1)^{\ell(w)}\prod_{\alpha\in\phi^+}(\delta,\alpha)$

$$= \boxed{\mathrm{sgn}(w)\prod_{\alpha\in\phi^+}(\delta,\alpha)}$$

Thus $\boxed{\mathrm{eval}(D(Q))} = \sum_{w\in W}\mathrm{sgn}(w)\,\boxed{\mathrm{eval}(D(e^{w\delta}))} = \sum_{w\in W}\mathrm{sgn}(w)^2\prod_{\alpha\in\phi^+}(\delta,\alpha)$

$$= \boxed{|W|\cdot\prod_{\alpha\in\phi^+}(\delta,\alpha)}$$

Similarly we derive $\mathrm{eval}(D(P)) = \sum_{w \in W} \mathrm{sgn}(w)\, \mathrm{eval}(D(e^{w(\lambda+\delta)}))$

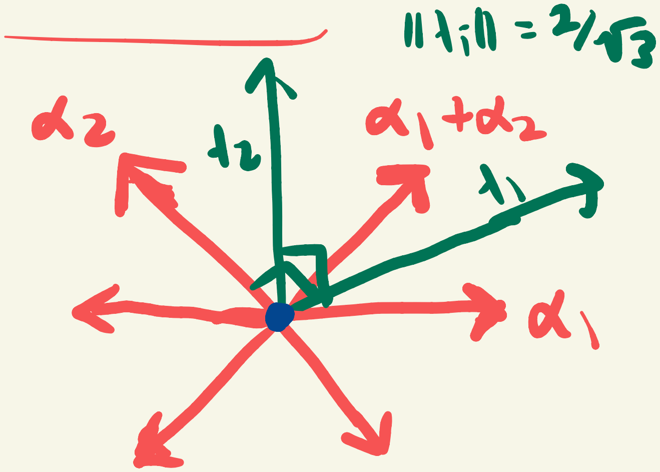
$= |W| \cdot \prod_{\alpha \in \Phi^+} (\delta+\lambda, \alpha)$. Thus, as $\deg(\lambda) = \dfrac{\mathrm{eval}(D(P))}{\mathrm{eval}(D(Q))}$:

<u>Cor</u> (Weyl dimension formula) If $\lambda \in \Lambda^+$ then

$$\deg(\lambda) \overset{\text{def}}{=} \dim V(\lambda) = \prod_{\alpha \in \Phi^+} \frac{(\delta+\lambda, \alpha)}{(\delta, \alpha)} = \prod_{\alpha \in \Phi^+} \frac{\langle \delta+\lambda, \alpha \rangle}{\langle \delta, \alpha \rangle}$$

where $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and $\langle \beta, \alpha \rangle = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$

Example (type A_2) Consider root system



Positive roots are $\alpha_1, \alpha_1+\alpha_2, \alpha_2$, and $\delta = \alpha_1+\alpha_2$

$(\vec{x}, \vec{y}) = \|\vec{x}\| \, \|\vec{y}\| \cos(\text{angle between } \vec{x} \text{ and } \vec{y})$ so

$$(\alpha_1, \alpha_1) = 1 = (\alpha_2, \alpha_2)$$

$$(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -1/2$$

$$(\alpha_1, \delta) = (\alpha_2, \delta) = \tfrac{1}{2} \text{ and } (\alpha_1+\alpha_2, \delta) = 1$$

Let $\lambda_1, \lambda_2 \in \mathbb{R}^2$ be such that

$$\langle \lambda_i, \alpha_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow \langle \delta, \alpha_1 \rangle = \langle \delta, \alpha_2 \rangle = 1, \ \langle \delta, \alpha_1+\alpha_2 \rangle = 2$$

$$\Rightarrow \boxed{\prod_{\alpha \in \phi^+} \langle \delta, \alpha \rangle = 2}$$

Every weight $\lambda \in \Lambda^+$ can be written uniquely as $\lambda = m_1\lambda_1 + m_2\lambda_2$

As $\langle \lambda+\delta, \alpha_1 \rangle = m_1+1$, $\langle \lambda+\delta, \alpha_2 \rangle = m_2+1$, $\langle \lambda+\delta, \alpha_1+\alpha_2 \rangle = m_1+m_2+2$

we end up with $\boxed{\deg(\lambda) = \tfrac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)}$