MATH 5143 – Lecture 24

Overview: Chevalley algebras and groups

Keep same notation for $L, H, \Phi, \Delta, \Phi^+, W, \mathbb{F}$ etc.

Idea: there is a basis for L whose structure constants are all integers, so we can realize L as a Lie algebra generated by matrices over $\mathbb{Z}$.

By extending scalars, one can construct L and its representations over arbitrary fields (rather than just the nice field $\mathbb{F}$)

Recall that $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ where

$$L_\alpha \overset{\text{def}}{=} \{ x \in L \mid [h, x] = \alpha(h) x \ \forall h \in H \}.$$

Key properties: $\dim L_\alpha = 1$

$\dim H = |\Delta|$

$[L_\alpha, L_{-\alpha}]$ is a 1-dimensional subspace of H spanned by a certain element h_α

To be explicit: writing $\kappa(x, y) = \operatorname{trace}(\operatorname{ad} x \operatorname{ad} y)$, we can define $t_\alpha = \frac{1}{\kappa(x,y)} [x, y]$ for any nonzero $x \in L_\alpha$, $y \in L_{-\alpha}$. Then t_α is unique elem of H with $\kappa(t_\alpha, h) = \alpha(h)$. Then $h_\alpha = \frac{2 t_\alpha}{\kappa(t_\alpha, t_\alpha)}$

Def A Chevalley basis for L is a basis

$$\{x_\alpha \in L_\alpha \mid \alpha \in \Phi\} \sqcup \{h_1, h_2, \dots, h_n \in H\}$$

such that (a) $[x_\alpha, x_{-\alpha}] = h_\alpha \ \forall \alpha \in \Phi$

(recall, h_α is defined entirely from L, H, Φ)

only part we haven't seen before

(b) if $\alpha, \beta, \alpha+\beta \in \Phi$ and $[x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$ for $c_{\alpha\beta} \in \mathbb{F}$, then $c_{\alpha\beta} = -c_{-\alpha,-\beta}$

(c) Φ has a simple system Δ such that $\{h_1, h_2, \dots, h_n\} = \{h_\alpha \mid \alpha \in \Delta\}$

Prop L has a Chevalley basis, and the coefficients $c_{\alpha\beta}$ corresponding to any such basis satisfy

$$(*) \quad \boxed{c_{\alpha\beta}^2 = q(r+1)\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)}}$$

where the α-root string through β is

$$\beta - r\alpha,\ \beta - r\alpha + \alpha,\ \ldots,\ \beta + q\alpha$$

Pf idea Slightly technical but elementary algebraic exercise given what we know about the root space decomposition. $\square$

Thm If $\{x_\alpha \in L_\alpha \mid \alpha \in \Phi\} \cup \{h_1, h_2, \dots, h_n \in H\}$ is a Chevalley basis for L the corresponding structure constants are all integers:

(a) $[h_i, h_j] = 0 \quad \forall i, j$

(b) $[h_i, x_\alpha] = \underbrace{\langle \alpha, \alpha_i \rangle}_{\in \mathbb{Z}} x_\alpha$

[where $\Delta = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is simple system for Φ such that $h_i = h_{\alpha_i}$.]

(c) $[x_\alpha, x_{-\alpha}] = h_\alpha \in \mathbb{Z}\text{-span}\{h_1, h_2, \dots, h_n\}$

(d) if $\alpha, \beta \in \Phi$ are non-proportional then

$$[x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \Phi \\ \underbrace{\pm(r+1)}_{\in \mathbb{Z}} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \end{cases}$$

[where the α-root string through β is $\beta - r\alpha, \dots, \beta + q\alpha$]

Pf sketch We have $[h_i, h_j] = 0$ since H is abelian.

We have (by def) $[h_i, x_\beta] = \alpha(h_i)\, x_\beta = \dfrac{2\beta(t_{\alpha_i})}{\kappa(t_{\alpha_i}, t_{\alpha_i})} x_\beta$

$$= \frac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)} x \simeq \langle \beta, \alpha_i \rangle x$$

We have $[x_\alpha, x_{-\alpha}] = h_\alpha$ by defn of a Chevalley basis.
We want to show that $h_\alpha \in \mathbb{Z}\text{-span}\{h_1, h_2, \dots, h_n\}$. For this:
let $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ then $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is a root system with base $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$, under the killing form identification of H with H^*, $t_\alpha \leftrightarrow \alpha$ and $h_\alpha \leftrightarrow \alpha^\vee$, and each $\alpha^\vee$ is a $\mathbb{Z}$-linear comb. of $\Delta^\vee$ so each h_α is lin. comb. of $\{h_1, h_2, \dots, h_n\}$.

Final property that $[x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \Phi \\ \pm(r+1)\, x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \end{cases}$

follows from the proposition and this lemma:

$\boxed{\text{tells us that } c_{\alpha\beta} = (r+1)^2 \Longrightarrow c_{\alpha\beta} = \pm(r+1)}$

<u>Lemma</u> If $\alpha, \beta \in \Phi$ are nonproportional and the α-string through β is $\beta - r\alpha, \dots, \beta + q\alpha$ then

$$\boxed{r+1 = \frac{q(\alpha+\beta, \alpha+\beta)}{(\beta,\beta)}}$$

<u>Pf</u> idea: either case-by-case argument, considering rank two root systems, or see uniform geometric argument in text book. $\square$

About uniqueness: given $L \supset H$, Φ, etc., a Chevalley basis is "almost unique": once a simple system $\Delta \subset \Phi$ is chosen, the h's are determined, but there is some flexibility in constructing the x_α's.

Fix a Chevalley basis for L

Let $L(\mathbb{Z}) \overset{\text{def}}{=} \mathbb{Z}\text{-span}[\text{this Chevalley basis}]$

$= \mathbb{Z}\text{-span}[x_\alpha \ (\alpha \in \Phi),\ h_1, h_2, \ldots, h_n]$

This is "Lie algebra over $\mathbb{Z}$" in the obvious sense, inheriting Lie bracket from L.

For a prime finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ ($p > 0$ prime)

can define $L(\mathbb{F}_p) \overset{\text{def}}{=} L(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$

More generally, for any field extension $\mathbb{K} \supseteq \mathbb{F}_p$ can define

$L(\mathbb{K}) \overset{\text{def}}{=} L(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{K} = L(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}$

Fact Then $L(\mathbb{K})$ is a Lie algebra over $\mathbb{K}$.

Call this a **Chevalley algebra**.

How do the tensors $\otimes_{\mathbb{Z}}$ work?

You have symbols $a \otimes_{\mathbb{Z}} b$ but not for any $k \in \mathbb{Z}$ we have $ka \otimes_{\mathbb{Z}} b = a \otimes_{\mathbb{Z}} kb$, plus usual tensor formalities.

Exercise Isomorphism class of $L(\mathbb{K})$ depends only on L not on the choice of Chevalley basis.

<u>Ex</u> If $L = \mathfrak{sl}_n(\mathbb{F})$ then $L(\mathbb{K}]$ has same multiplication table relative to usual standard basis, so $L(\mathbb{K}) \cong \mathfrak{sl}_n(\mathbb{K})$. Only change in this case is that $\mathfrak{sl}_n(\mathbb{K})$ may no longer be simple if char$(\mathbb{K})$ divides n.

<u>Prop</u> Let $\alpha \in \Phi$ and $m \in \mathbb{Z}_{\geq 0}$. Then $(\mathrm{ad}\, x_\alpha)^m / m!$ preserves $L(\mathbb{Z})$. <u>Pf</u> straightforward calculation $\square$

Def Express $\mathrm{ad}\, x_\alpha$ as a (nilpotent) matrix M_α relative to the Chevalley basis.

Define $$x_\alpha(t) = \exp(tM_\alpha) = \underbrace{\sum_{k\geq 0} \frac{t^k}{k!} M_\alpha^k}_{\text{finite sum}}$$

The Chevalley group (of adjoint type) is then

$$G(\mathbb{K}) \stackrel{\text{def}}{=} \langle x_\alpha(t) \mid \alpha \in \Phi,\ t \in \mathbb{K} \rangle \subset GL(L(\mathbb{K}))$$

Key facts about Chevalley groups

no proper nontrivial normal subgroups

1. When $\mathbb{K}$ is finite the group $G(\mathbb{K})$ is finite
2. When $\mathbb{K}$ is finite and L is simple, $G(\mathbb{K})$ is a finite $\boxed{\text{simple group}}$ outside a short list of exceptional cases.
3. There is a uniform way of proving that these Chevalley groups are almost always simple, and this gives most of the infinite families of finite simple groups.
4. As usual, isomorphism type of $G(\mathbb{K})$ depends on L but not on the choice of Chevalley basis.