MATH6380L: Mathematical foundations of imaging
Fall 2017

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Lecture: Tue Thu 04:30PM-05:50PM, Room 5510 (Lift 25-26).

Main references:

1. Medical Imaging: Signals and Systems, 2nd Edition, Jerry L. Prince and Jonathan Links, Pearson, 2015.
2. Introduction to the Mathematics of Medical Imaging: Second Edition, Charles L. Epstein, SIAM, 2007.
3. Foundations of Medical imaging, Zang-Hee Cho, Joie P. Jones and Manbir Singh, Wiley, 1993.
4. Image processing and analysis: variational, PDE, wavelet, and stochastic methods, Tony F. Chan, Jackie Shen, SIAM, 2005
5. A mathematical introduction to compressive sensing, Simon Foucart, Holger Rauhut, Birkhauser, 2013.
6. Convex optimization, Stephen Boyd, Lieven Vandenberghe, Cambridge University Press, 2004.
7. A Wavelet Tour of Signal Processing: The Sparse Way (Third Edition), Stephane Mallat, Academic Press, 2008.

Other resources: research papers, lecture notes of Math 262 (Applied Fourier Analysis and Elements of Modern Signal Processing) by Prof. E. Candes.

Course evaluation: Each student is required to complete at least one project and deliver an oral 30-mins presentation.

Course description and objective: this course aims to introduce the basic mathematical tools and techniques used in the field of imaging and medical imaging, which is becoming increasingly important nowadays. The following topics shall be covered: 1. the mathematical modelling of imaging modality; 2. the inversion step where measurements are used to generate the imaging; 3. image quality analysis and processing. Emphasize is given to the image quality analysis such as resolution and artifacts and their relations to the mathematical modeling, sampling, inversion scheme and signal to noise level. The course also serves to provide an introduction to a broad range of applied mathematical tools which are not covered in usual math curriculums such as sampling theory, filter theory, wavelets, approximation theory, convex optimization and compressive sensing. It is expected that the course will introduce the students to the frontiers of the research field of imaging and inverse problems.

The course will provide a comprehensive understanding to the imaging reconstruction for the linear modalities: CT imaging and MRI imaging. For the other nonlinear modalities, basic physical principles and mathematical models will be introduces and students are encourage to apply the methods and algorithms learned from the previous lectures to implement the reconstruction and carry out some mathematical analysis.

The presentation will focus on the key mathematical idea and avoid technical details, advanced and related topic will be assigned as projects. Students with a basic background of multi-variable calculus and linear algebra will be able to follow. The main content of the course is listed below.

1. Fourier transform
   Fourier transform and examples, inverse Fourier transform, regularity and decay, L^2 theory, Heisenberg uncertainty principle, Gibbs phenomenon.

2. Convolution and Shift invariant filters:
   Convolution and regularity, approximation by convolution, δ-function, Resolution, The inverse filters, Resolution of filters, Filters for periodic inputs, Hilbert transform.

3. Fourier series:
   Fourier inversion formula, Decay of Fourier coefficient and regularity, L^2-theory, convolution, partial sum approximation, Gibbs phenomenon, Resolution of partial sums.

4. Sampling theory:
   Nyquist's theorem, Shannon-Whittaker interpolation, Poission summation formula, Undersampling and aliasing, Subsampling, Finite Fourier transform, higher-dimensional sampling.

5. Implementing shift invariant filters:
   Finite Fourier transform, approximation of Fourier coefficients, approximation of periodic convolutions, implementing filters on finitely sampled data, zero padding, higher dimensional filters, FFT.

6. The Randon transform:
   Randon transform and inversion formula, Filtered back-projection, approximate inverse of Radon transform, well-posedness of Randon transform, (X-ray transform).

7. X-ray tomography and direct reconstruction
   History and physics background, mathematical formulation, Numerical reconstruction for Parallel beam scanner(Direct Fourier inversion, Filtered Back-Projection, Sampling spacing), Numerical reconstruction for Fan beam case, (Sprial scanner CT).

8. Artifacts analysis in X-ray tomography
   Effect of Beam finite-width, point spread function, artifact from ray sampling, artifact from view sampling, effect of measurement errors, effect of beam hardening.

9. Iterative method for solving linear systems
   Formulation of linear equations for CT imaging, Landweber iteration, Kaczmart iteration(algebraic reconstruction techniques), Krylov subspace method.

10. Probabilistic model and noise analysis
    Probabilistic models for X-ray generation and detection, Beer's law, Propagation of noise through back-projection algorithm, Signal to noise ratio and resolution.

11. Ill-posed problems and regularization
    Ill-posedness, Picard' criterion, Singular Value Decomposition, Tikhonov-regularization, Discrepancy principle, Equivalence of regularization and constrained optimization.

    END of Fall semester ———————————————-

    Intended for the Spring 2018, MATH6380

12. Basic of image processing
    Image modeling and Representation, image denosing; image deblurring; image inpainting; image segmentation.

13. Wavelet theory and multi-resolution analysis
    Continuous Wavelet Transform, Multi-Resolution Analysis, construction of Orthonormal Wavelets, Discrete Wavelet transform, non-linear approximation under wavelet expansion,

14. Convex analysis
    Convex sets, Convex functions, Duality, Sub-gradients, proximity operator

15. Convex optimization and algorithms
    Iterative thresholding algorithms, augmented Lagrangian algorithms, proximal algorithms, primal-dual algorithms.

16. Compressive sensing
    Compressed sensing, sparse solution of linear systems, L1-norm minimization, Restricted isometric property (RIP), Probabilistic construction of matrices satisfying RIP.

17. Magnetic resonance imaging
    Basics of Nuclear magenetic resonance, The Bloch phenomological equation, Selective excitation, Relaxation process, Basic signal equation for magnetic resonance imaging, Contrast, signal to noise ration and resolution, Basics of chemical shift imaging.

18. Phase retrieval problem
    Convex relaxation algorithms, PhaseLift, Non-convex methods, Wirtinger flow algorithms, geometry of non-convex optimization.

19. Ultrasound and image formation
    The acoustic wave equation, plane waves, spherical waves, The Green's function, Reflection and refraction at interfaces, attenuation, scattering, Doppler effect, Beam pattern formation and focusing (Fresnel-Kirchhoff diffraction formula, Rayleigh-Sommerfeld diffraction formula, Fresnel diffraction, Fraunhofer diffraction).

20. Resolution limit and super-resolution
    Abbe's diffraction limit, The inverse source problem, the time-reversal imaging method, numerical super-resolution and physical super-resolution, super-resolution techniques (near field, Single molecule imaging, structured illumination, super-lens).

21. Ultrasound imaging
    Basic Physical setup of ultrasound imaging system, Pulse-echo equations, augmentations to the basic imaging model, noise and resolution.

22. Multi-wave imaging
    Physics of photo-acoustic effect, mathematical formulation of photo-acoustic imaging, review of theoretical results, Discrete model and reconstruction.