Research Statement

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1. Introduction

To any finite dimensional complex simple Lie algebra $\mathfrak{g}$, Drinfeld [Dr85] and Jimbo [Ji85] associated the so-called **quantum group** $\mathcal{U}_q(\mathfrak{g})$. Originally motivated from the search for solutions to the *Yang-Baxter equation* important in *quantum integrable systems*, its representation theory has evolved into an important area with many applications to various fields of mathematics and physics since its discovery 30 years ago, which include:

(1) **3-manifolds and knot invariants** giving *3d topological quantum field theories* [RT90, RT91, Wi89] constructed through the *braided tensor category structure*;
(2) **Kazhdan-Lusztig theory** [KL93, KL94] of the equivalence of categories of highest weight representations of affine Lie algebra $\widehat{\mathfrak{g}}$ and quantum group $\mathcal{U}_q(\mathfrak{g})$.
(3) **Categorification** of $\mathcal{U}_q(\mathfrak{g})$ and its finite-dimensional representations giving rise to *Khovanov homology* [Kh99] and *Nakajima's quiver varieties* [Na94] through the use of *Lusztig's canonical basis* [Lu90];

In classical Lie theory, one is interested in certain real subalgebras of $\mathfrak{g}$ known as *real forms*, two important cases being $\mathfrak{g}_c$ corresponding to **compact** groups (e.g. $SU(n)$), and $\mathfrak{g}_\mathbb{R}$ corresponding to **split real** groups (e.g. $SL(n,\mathbb{R})$). The finite-dimensional representation theory in the compact case is well-behaved, and it is generalized nicely to the corresponding quantum group $\mathcal{U}_q(\mathfrak{g}_c)$. On the contrary, representation theory in the split real case is much more complicated as was shown by the monumental works of Harish-Chandra. Its generalization to the quantum group level – involving self-adjoint operators on Hilbert spaces – is physically more relevant, but is still largely open due to various analytic difficulties coming from non-compactness and the use of unbounded operators.

In my joint work with Igor Frenkel [FI14], we introduced the theory of **positive representations** as a new research program to study the representation theory of **split real quantum groups** $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$, which is further developed in my works [Ip12a-Ip16]. These representations are natural generalizations of a special class of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ studied by Teschner *et al.* [BT03, PT99, PT01] from the physics point of view, which is characterized by the actions of *positive* self-adjoint operators on $L^2(\mathbb{R})$.

Surprisingly, in contrast to the complicated situation of classical real group, this class of representations for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ possesses nice structural properties parallel to the compact case! Most importantly, it again carries certain *braided tensor category* structure, and the *harmonic analysis* also gives a Peter-Weyl type theorem [Ip13b].

On the other hand, the split real case $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ exhibits many new phenomena which are not present in the compact case. To name a few, we have Faddeev's **modular double** [Fa95, Fa99], connection to **quantum Teichmüller theory** [Te07, FK12] and **cluster algebra** [BZ05, FG09], the use of the remarkable **quantum dilogarithm** [FaK94], and the simplest occurrence of **Langlands duality** as functional analytic relations [Ip15b].

The successful construction of positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$, together with the nice properties enjoyed by those of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ resembling the compact case, strongly indicates that many applications including (1)–(3) of the finite dimensional representation theory of $\mathcal{U}_q(\mathfrak{g}_c)$ can be generalized to $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$. This will be a new advancement in mathematics, and I envision it as the future perspective of my research program.

Below I will give an overview of the theory of positive representations and some known results established by myself and others, and elaborate in more details several research directions that I am actively pursuing, including

- Generalization of the braided tensor category structure to higher rank;
- Cluster algebraic realization of positive representations;
- Construction of quantum higher Teichmüller theory; and
- Harmonic analysis and classical limit of positive quatum groups.

2. Positive representations of split real quantum groups

The notion of the *positive principal series representations*, or simply *positive representations*, was introduced in [FI14] as a new research program devoted to the representation theory of split real quantum groups $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$.

Let $q = e^{\pi i b^2}$ be on the unit circle that is not a root of unity. We denote by $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ the Hopf-* algebra of Drinfeld-Jimbo type [Dr85, Ji85] associated to simple Lie algebra $\mathfrak{g}$ generated by $\{E_i, F_i, K_i\}_{i\in I}$, where I is the set of simple roots, such that the generators are self-adjoint:

$$E_i^* = E_i, \qquad F_i^* = F_i, \qquad K_i^* = K_i. \tag{1}$$

In the work of Teschner *et al.* [BT03, PT99, PT01], a special class of representations $\mathcal{P}_\lambda$ of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, parametrized by $\lambda \in \mathbb{R}_+$, has been studied through the fusion relations of *quantum Liouville theory*, which is the quantization of the simplest non-compact conformal field theory constructed from certain highest weight representations of the *Virasoro algebra.* Here the generators are represented by positive self-adjoint operators acting on $L^2(\mathbb{R})$, and the action has *no classical limit.* This is explained by Faddeev [Fa95, Fa99] who proposed that in the split real setting, it is natural to consider also its *modular double*, by raising the generators to a transcendental power:

$$\widetilde{X} := X^{\frac{1}{b^2}}, \tag{2}$$

which make sense since the generators are positive, and commute weakly with the original generators. Let $\widetilde{q} = e^{\pi i b^{-2}}$. Then we denote by $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R}) := \mathcal{U}_q(\mathfrak{g}_\mathbb{R}) \otimes \mathcal{U}_{\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ the split real quantum group together with its modular double counterparts.

We generalize this class of representations to higher rank as follows [Ip12, Ip15b].

Theorem 2.1. *There exists a family of representations $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ such that*

- *$\mathcal{P}_\lambda$ is parametrized by $\lambda \in \mathbb{R}_+$-span of the positive weights $P_+ \subset \mathfrak{h}_\mathbb{R}^*$.*
- *$\{E_i, F_i, K_i\}$ are represented by positive, essentially self-adjoint unbounded operators on $L^2(\mathbb{R}^{\dim(U_+)})$, where U_+ is the upper unipotent subgroup.*
- *Define (for rescaled generators)*

$$\widetilde{\boldsymbol{e}}_i := \boldsymbol{e}_i^{\frac{1}{b^2}}, \qquad \widetilde{\boldsymbol{f}}_i := \boldsymbol{f}_i^{\frac{1}{b^2}}, \qquad \widetilde{K}_i := K_i^{\frac{1}{b^2}} \tag{3}$$

  – *in the simply-laced case, $\{\widetilde{\boldsymbol{e}}_i, \widetilde{\boldsymbol{f}}_i, \widetilde{K}_i\}$ generates $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_\mathbb{R})$,*

  – *in the non-simply-laced case, $\{\widetilde{\boldsymbol{e}}_i, \widetilde{\boldsymbol{f}}_i, \widetilde{K}_i\}$ generates the Langlands dual $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_\mathbb{R})$.*

- *the generators $\{\widetilde{\boldsymbol{e}}_i, \widetilde{\boldsymbol{f}}_i, \widetilde{K}_i\}$ commute weakly with $\{\boldsymbol{e}_i, \boldsymbol{f}_i, K_i\}$ up to a sign.*

The construction uses the theory of *Lusztig's total positivity* [Lu94]. As a corollary, for the first time it gives an explicit generalization of *Feigin's homomorphism* [Re01] of the upper half of the quantum group $\mathcal{U}_q(\mathfrak{b})$ to the whole quantum group $\mathcal{U}_q(\mathfrak{g})$, embedding it into certain quantum torus algebra. This requires substantial amounts of combinatorics involving the remarkable *quantum dilogarithm* $g_b(x)$, which plays a central role in the technical backbone of the split real theory.

In the case of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, it has several properties that is in complete parallel to the representation theory of compact quantum groups, namely the existence of a universal R matrix, and closure under taking tensor products:

Proposition 2.2. [BT03] *There exists a unitary operator R acting on $L^2(\mathbb{R}\times\mathbb{R})$ given by*

$$R = q^{\frac{H\otimes H}{4}} g_b(\boldsymbol{e}\otimes\boldsymbol{f}) q^{\frac{H\otimes H}{4}} \tag{4}$$

such that it satisfies the braiding relation

$$R\Delta(X) = \Delta^{op}(X)R \tag{5}$$

for $X\in\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$, and it is invariant under the change $b\longleftrightarrow b^{-1}$. Here we formally write $K=q^H$ as operators, and $g_b(x)$ is the quantum dilogarithm function.

Theorem 2.3. [PT01] *$\mathcal{P}_\lambda$ is closed under taking tensor product (in a continuous sense). We have*

$$\mathcal{P}_\alpha\otimes\mathcal{P}_\beta \simeq \int_{\mathbb{R}_+}^{\oplus} \mathcal{P}_\gamma d\mu(\gamma) \tag{6}$$

where $d\mu(\gamma)$ is expressed in terms of quantum dilogarithm.

Together, these give certain "continuous" braided tensor category structure to the positive representations of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, and it is expected that we can obtain new *topological quantum field theory (TQFT)* in the sense of Reshetikhin-Turaev [RT90, RT91, Wi89].

Therefore it is natural to extend this to higher rank, and we can try to find the analogues of various remarkable results and constructions that were discovered and studied in relation to the braided tensor category of the finite-dimensional representations of the compact quantum group $\mathcal{U}_q(\mathfrak{g}_c)$.

First of all, we have the braiding structure from an analogue of universal R matrix:

Theorem 2.4. [Ip15a] *There exists a unitary operator R giving the braiding structure*

$$\mathcal{P}_\mu\otimes\mathcal{P}_\lambda \simeq \mathcal{P}_\lambda\otimes\mathcal{P}_\mu \tag{7}$$

for the positive representations of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$.

The universal R-operator is constructed for $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ following the compact case by [KR90, LS90], but utilizes a continuous version of PBW basis of a C^*-algebraic version of the Borel part $\mathcal{U}_{q\widetilde{q}}^{C^*}(\mathfrak{b}_\mathbb{R})$ in the sense of multiplier Hopf algebra introduced in [vD94]. Therefore the remaining important problem is to find the tensor category structure:

Problem 2.5. *Show that the class of positive representations $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ is closed under taking tensor products.*

In the next section, we will describe several progresses towards the construction of the tensor category structure.

3. Generalization of braided tensor category to higher rank

One important problem for positive representations is the construction of the tensor category structure. Some progresses have been made recently, which we describe below.

Positive Casimir operators. In the case of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, the closure under tensor products is proved in [NT13] by studying the spectrum of the Casimir operator $C := FE + \left[H + \frac{1}{2}\right]_q^2$. Therefore it may be important to understand the correspond central elements in higher rank. In [Ip16], we utilized the explicit construction from [BGZ91] in the compact case with the R matrix to construct the corresponding Casimir operators for the positive representations. Using a new notion of *virtual highest weights*, we found closed formulas for the eigenvalues of the Casimir operators as a sum of exponentials, which in particular shows that the central elements have positive spectrums. As a corollary, we showed that their images define a semi-algebraic region bounded by real points of the so-called *discriminant variety* with a singularity point, which potentially give new explicit relations among quantum group theory, simple (du Val) singularities and the theory of primitive forms [Sai93, Sai04].

$$C_1(\lambda) = e^{\frac{8}{3}\pi b\lambda_1 + \frac{4}{3}\pi b\lambda_2} + e^{-\frac{4}{3}\pi b\lambda_1 + \frac{4}{3}\pi b\lambda_2} + e^{-\frac{4}{3}\pi b\lambda_1 - \frac{8}{3}\pi b\lambda_2}$$
$$C_2(\lambda) = e^{\frac{4}{3}\pi b\lambda_1 + \frac{8}{3}\pi b\lambda_2} + e^{\frac{4}{3}\pi b\lambda_1 - \frac{4}{3}\pi b\lambda_2} + e^{-\frac{8}{3}\pi b\lambda_1 - \frac{4}{3}\pi b\lambda_2} \longrightarrow$$

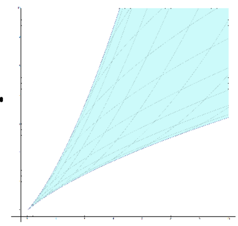

Figure 1. Spectrum of positive Casimirs of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(3,\mathbb{R}))$ as A_2 singularity

Analytic continuation and continuous canonical basis. Recently in [Ip15d], we studied in detail the correspondence between the tensor product decomposition of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ and its compact counterpart $\mathcal{U}_q(\mathfrak{sl}_2)$. We show that by solving certain functional equations and using normalization arising from tensor products of *canonical basis*, we can derive the Hilbert space decomposition from Theorem 2.3, given explicitly by quantum dilogarithm transformations, from the Clebsch-Gordan coefficients of the tensor product decomposition of finite dimensional representations of the compact quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. In particular, we observe that in fact the structural constants of positive representations appear as some *twisted analytic continuation* of the compact case. This allows us to propose a general strategy to deal with tensor product decomposition for higher rank split real quantum group $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$. Roughly speaking, one should solve explicitly the decomposition for $\mathcal{U}_q(\mathfrak{g}_c)$ in terms of q-factorials, replace the summation by appropriate integrals of quantum dilogarithm functions, and normalize correspondingly.

On the other hand, using the continuous PBW basis introduced in the previous section, we showed in [Ip14] that the positive representations restricted to the Borel part is actually closed under tensor product. From the nice behaviors of the canonical basis with tensor products decomposition, we can try to modify such continuous PBW basis and search for the notion of *continuous canonical basis* [Ip13c] that is believed to provide the answer to the main problem.

4. Embedding to quantum cluster algebra

Recall that the construction of positive representations gives for the first time an explicit generalization of Feigin's homomorphism from the Borel part $\mathcal{U}_q(\mathfrak{b})$ to the whole quantum group $\mathcal{U}_q(\mathfrak{g})$, embedding it into certain quantum torus algebra. In particular the use of Lusztig's total positivity and quantum dilogarithm transformation suggests a strong connection of positive representations to quantum cluster algebra.

Recently in [SS16], it was shown that the quantum group $\mathcal{U}_q(\mathfrak{sl}_{n+1})$ admits an algebraic embedding into certain quantum cluster $\mathcal{X}$-variety in the sense of [FG06, FG09] associated to a decorated punctured disk. It turns out that this embedding is identical to the quantum torus realization of the positive representations for type A_n, and we discovered that using the explicit construction of positive representations, one can actually generalize the type A_n case and find the embeddings for *all finite types* quantum groups. The quantum cluster varieties in these cases are realized by certain newly discovered quivers associated to triangulations of the punctured disk. Furthermore, we computed in some cases that one can decompose the R-matrix explicitly into products of quantum dilogarithms. These correspond to a mutation sequence of two copies of the quiver, which gives exactly the 4 flips of quadrilateral that is required to produce the half Dehn twist, realizing the braiding structure of the embedding. We conjecture that this is true for all types of $\mathcal{U}_q(\mathfrak{g})$. These are partially in joint work with G. Schrader and S. Shapiro.

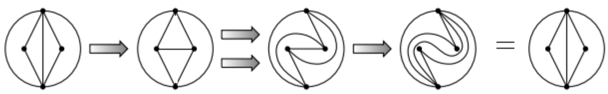

Figure 2. Mutation giving half Dehn twist

It turns out that the quivers associated to our quantum torus algebra are mutation equivalent to the recent discovery of the cluster structures on the moduli space of (classical) local systems by [Le16] for general groups. Thus our results quantize Le's construction of the cluster structure, which provides an important candidate of the construction of quantum higher Teichmüller theory described in the next section.

5. Quantum higher Teichmüller Theory

The *Teichmüller space* $\mathcal{T}_S$ of a surface S is the space of all complex structures on S modulo diffeomorphisms isotopic to identity. It is a very important space closely related to the moduli space of Riemann surfaces, and carries a natural action of the *mapping class group* Γ_S, i.e., the group of all orientation-preserving diffeomorphisms modulo isotopy, that preserves a canonical Poisson structure called the *Weil-Petersson form*.

Hence, quantum Teichmüller theory is roughly speaking the quantization $\mathcal{T}_S^q$ of the Poisson manifold $\mathcal{T}_S$, such that the non-commutative algebra of function is represented on some Hilbert space of states $\mathcal{H}$, and automorphisms of $\mathcal{T}_S^q$ associated to $g \in \Gamma_S$ is represented by unitary operators $\rho(g)$ on $\mathcal{H}$. Thus, the **main goal** is to construct new projective unitary representations of the mapping class group Γ_S.

On the other hand, the space $\mathcal{T}_S$ can alternatively be described as certain component of the moduli space of $SL(2,\mathbb{R})$-local systems on S. Therefore the notion of quantum higher Teichmüller theory can roughly be considered as the study of the quantization of the modular space of $G(\mathbb{R})$ local systems [FG06, FG07].

In the previous section we have seen how one can use positive representations to give a quantization based on Le's discovery of the cluster structures of the local systems. In this section, we will describe another approach based on the generalization of Frenkel-Kim's construction from the representation theoretic point of view. However, in view of [Kim16a], the two approaches are most likely not equivalent to each other, and it will be interesting to understand the connections between the two.

Frenkel-Kim's construction. The quantum Teichmüller space was constructed in [FK12] from the representation theory of the modular double of the quantum plane $\mathcal{B}_{q\tilde{q}}$, which is just the restriction $\mathcal{H}$ of the positive representations to the Borel part of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. It was shown that $\mathcal{H}$ is closed under taking the tensor product and decomposes as

$$\mathcal{H}\otimes\mathcal{H}\simeq M\otimes\mathcal{H} \tag{8}$$

where $M\simeq \mathrm{Hom}_{\mathcal{B}_{q\tilde{q}}}(\mathcal{H},\mathcal{H}\otimes\mathcal{H})$ is the multiplicity module with $\mathcal{B}_{q\tilde{q}}$ acting trivially. Upon identification of $\mathcal{H}$ and M with $L^2(\mathbb{R})$, the canonical isomorphism

$$(\mathcal{H}_1\otimes\mathcal{H}_2)\otimes\mathcal{H}_3\simeq\mathcal{H}_1\otimes(\mathcal{H}_2\otimes\mathcal{H}_3) \tag{9}$$

yields an operator $\mathbf{T}$ on the multiplicity modules, called the *quantum mutation operator*, and by construction satisfies the pentagon relation. On the other hand, the identification of left and right dual representations through the antipode and its inverse:

$$'\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3\simeq\mathrm{Hom}(\mathcal{H}_1,\mathcal{H}_2\otimes\mathcal{H}_3)\simeq\mathcal{H}_2\otimes\mathcal{H}_3\otimes\mathcal{H}_1' \tag{10}$$

induces another operator $\mathbf{A}$ where $\mathbf{A}^3=1$. Together they recover Kashaev's projective representation of the mapping class groupoid $\mathfrak{G}$ associated to dotted triangulations of the surface S, and we can apply it to quantization of Teichmüller space [CF99, Ka98].

Higher theory. Generalizing to higher rank, a natural candidate is to replace $\mathcal{H}$ by the restriction of the positive representations of more general split real quantum groups $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ to its Borel subalgebra $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_\mathbb{R})$. It follows that the main ingredient needed to construct a version of quantum higher Teichmüller theory is the construction of the quantum mutation operator $\mathbf{T}$ as described above, coming from the decomposition of tensor products of positive representations of the Borel part, as well as the $\mathbf{A}$ operator coming from the dual representations.

In [Ip14], the operator $\mathbf{T}$ is constructed by studying the tensor products of $\mathcal{P}_\lambda$ restricted to the Borel part. We used the theory of continuous PBW basis to construct the appropriate C^*-algebraic version of the Borel subalgebra, which is shown to be independent of the parameter λ, and therefore we can apply the theory of multiplicative unitary from multiplier Hopf algebra to construct the required unitary equivalence as in (8).

Therefore we try to solve the remaining problems:

Problem 5.1. *Construct the Kashaev's* $\boldsymbol{A}$ *operator.*

Problem 5.2. *Show that* $\boldsymbol{A}$ *and* $\boldsymbol{T}$ *are compatible with each other, i.e. they satisfy both the consistency and inversion relations* [Ka98, Kim16b]*.*

Furthermore, a version of Kashaev's ratio coordinates is recently constructed for higher Teichmüller spaces [Kim16b]. Therefore it is also useful to relate the two approaches:

Problem 5.3. *Understand the relationship between the quantization by positive representations of the Borel part, and the appropriate quantization of Kashaev's ratio coordinates for higher Teichmüller space.*

6. Harmonic analysis and classical limits of positive quantum groups

Peter-Weyl Theorem. For a compact group G, we know that the functions on G can be decomposed into the finite dimensional regular representation of $U(\mathfrak{g})$

$$Fun(G)=\bigoplus_\lambda V_\lambda\otimes V_\lambda^*, \tag{11}$$

parametrized by the positive weights $\lambda\in P^+\subset\mathfrak{h}_\mathbb{R}^*$. For split real group $G_\mathbb{R}$ this becomes more complicated. For example if $G=SL(2,\mathbb{R})$, we know that the regular representation of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))$ on $L^2(SL(2,\mathbb{R}))$ is decomposed into both the continuous and discrete series. However, in the quantum split real case, a Peter-Weyl type theorem is proposed in [PT99] and proved in [Ip13b], which surprisingly is in complete parallel to the compact case:

Theorem 6.1. [Ip13b] *We have the following unitary equivalence*

$$L^2(SL_{q\widetilde{q}}^+(2,\mathbb{R}))\simeq\int_{\mathbb{R}_+}^{\oplus}P_\gamma\otimes P_\gamma d\mu(\gamma) \tag{12}$$

as a $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ left and right regular representation.

Here $\mathcal{P}_\gamma$ is the positive representations, $L^2(SL_{q\widetilde{q}}^+(2,\mathbb{R}))$ is a Hilbert space constructed from the GNS representation of a C^*-algebraic version of the modular double of the positive quantum group $SL_q^+(2,\mathbb{R})$, which is also a locally compact quantum group in the sense of [KV00]. The action of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ is then obtained by dualizing the regular corepresentation of $SL_q^+(2,\mathbb{R})$. Finally, the Plancherel measure is exactly the same as in the tensor product decomposition (6), expressed in terms of the quantum dilogarithm.

A natural problem is to extend this result to higher rank:

Conjecture 6.2. *Show that the regular representation of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ on the space $L^2(G_{q\widetilde{q}}^+(\mathbb{R}))$ decomposes into direct integral of positive representations.*

The Hilbert space $L^2(GL_{q\widetilde{q}}^+(n,\mathbb{R}))$ for type A_{n-1} is constructed using the *Gauss-Lusztig decomposition* for the positive quantum group $GL_q^+(n,\mathbb{R})$ introduced in [Ip15c]. General type construction should follow by dualizing the quantum torus algebra embedding of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ discussed in Section 4, which is compatible with the modular double and preserves positivity, hence allows us to define general $G_{q\widetilde{q}}^+(\mathbb{R})$ in the C^*-algebraic setting.

Limit of positive quantum groups. It is natural to ask if the above results in the quantum split real case descend to new structural results for classical semigroups. Although the positive representations $\mathcal{P}_\lambda$ as well as the quantum dilogarithm g_b has no direct classical limit, it was shown in [Ip13a] that g_b tends to the classical Gamma function $\Gamma(x)$ under certain rescaled limit. Using this fact, we found for example in [Ip13b] that the matrix coefficients of the fundamental corepresentation of $SL_q^+(2,\mathbb{R})$ tends to the hypergeometric functions ${}_2F_1(z)$ under the classical limit, and the resulting formulas give precisely the expressions corresponding to the principal series representations restricted to the positive semigroup $SL^+(2,\mathbb{R})$. We also observed new representation theoretic meaning of the *beta pentagon equation* [Ka14] satisfied by $\Gamma(x)$ coming from the limit of the representation of the quantum plane [Ip13a].

On the other hand, recently in [IY15], we studied the limit of the quantum dilogarithm when q goes to root of unity. In particular the case when $q\longrightarrow-1$ is very interesting as it recovers the classical $ax+b$ group on one hand, while the quantum dilogarithm tends to the classical dilogarithm function, which potentially gives new representation theoretic results even on the simplest rank 1 case.

Finally, we expect to gain more insight into physical questions by understanding the relationship between the quantum and classical picture. For example, we know that the classical semi-group is closely related to causality in physics [HN93]. On the other hand, the non-existence of classical limits of the positive representations of $U_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ suggested certain kind of renormalization in 4d QFT [BS59], that separates the discrete series representations of the classical group as discussed in [FaL07].

7. Further Research Directions

Super Teichmüller theory and quantum groups. Another generalization of Teichmüller theory of interest is to consider its super analogue. In a recent joint work with Penner and Zeitlin [IPZ16], we constructed new decorated super-Teichmüller theory in the $\mathcal{N}=2$ case corresponding to the structure group $OSp(2|2)$, generalizing the previous $\mathcal{N}=1$ case [PZ15] that is also related to the recent quantization constructed in [APT15].

As an application of positive representations, in joint works with Zeitlin, we studied the modular double structure for quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2,\mathbb{R}))$ [IZ14a] and the twisted form $C_q^{(2)}(2)$ [IZ14b]. Using a spinor trick, we constructed its representations and the corresponding Q-operator so that every objects can be expressed by the standard $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ together with an extra Clifford basis. As a consequence, this gives us immediately the rather complicated degree 4 super Casimir operators simply as the square of classical quadratic Casimir operator twisted by odd generators. It is interesting to continue this direction to establish the tensor category structure for the super case, much like the case of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, its generalization to higher rank, and quantization of the above super Teichmüller theory.

Toda field theory and W-algebra. The equivalence of categories of highest weight representations of affine Lie algebras and quantum groups were extensively studied in [KL93, KL94]. The explicit construction of the equivalence can be simplified by considering an additional category of representations of W-algebra [Sty98]. In fact, the original approach to the tensor product structure of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ is motivated from the fusion relations of the quantum Liouville theory [PT99, PT01]. There it was shown that the Racah-Wigner coefficients for triple tensor product decomposition is identical to the transformation between the s-channel and t-channel in the fusion relations of conformal blocks of Virasoro algebra, which is the W-algebra associated to $\mathfrak{sl}(2,\mathbb{R})$. Therefore one can try to understand positive representations in the higher rank from the point of view of the corresponding non-compact CFT's known as Toda conformal field theory [FaL07, Wy09] and its relationship with general W-algebra associated to $\mathfrak{g}_\mathbb{R}$. Finally, quantum Liouville theory is strongly related to quantum Teichmüller theory where the space of states of both theories are identified [Te07, Te14]. Therefore this provides another evidence for the relationship among positive representations of split real quantum groups, Toda field theory, and quantum higher Teichmüller theory.

Continuous Categorification. In the compact finite dimensional case, the highest weight representation V_λ is decomposed into direct sum of weight spaces $\oplus V_\lambda(n)$, and roughly speaking categorification is done by replacing these spaces with categories such that the quantum generators become functors between categories which satisfy certain conditions involving natural transforms. In the case of the positive representations, the representations are now infinite dimensional with continuous weight spaces. As proposed in [FI14], it will be very interesting to solve even for the simplest case:

Problem 7.1. *Find the notion of a "continuous categorification" for the split real quantum groups $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ and its family of positive representations $\mathcal{P}_\lambda$.*

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