The Mysterious Dilogarithm

Ivan Ip

Department of Mathematics
Yale University

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$\mathrm{Li}_2(x)$ is called the dilogarithm function.

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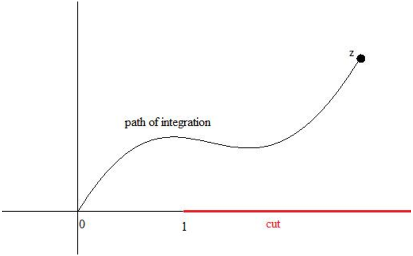
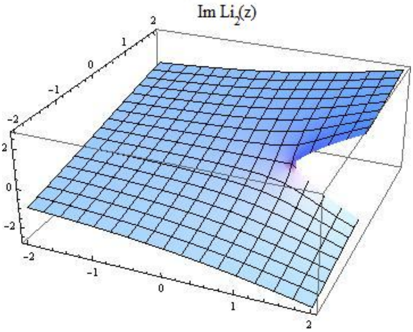
Hence we can give an analytic continuation of the dilogarithm by:

Definition

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u} \qquad \text{for } z \in \mathbb{C} \setminus [1, \infty)$$


z
path of integration
0
1
cut


Im $Li_2(z)$
2
1
0
-1
-2
2
0
-2
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0
1
2

Reflection properties

Proposition

$$\mathrm{Li}_2\left(\frac{1}{z}\right) + \mathrm{Li}_2(z) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z)$$

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Proof: Differentiating both sides.
Applying these formula, we see that the 6 functions:

$$\mathrm{Li}_2(z), \mathrm{Li}_2\left(\frac{1}{1-z}\right), \mathrm{Li}_2\left(\frac{z-1}{z}\right), -\mathrm{Li}_2\left(\frac{1}{z}\right), -\mathrm{Li}_2(1-z), -\mathrm{Li}_2\left(\frac{z}{z-1}\right)$$

are equal modulo elementary functions.

Duplication formula

Proposition (Duplication formula)

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and more generally the "distribution property":

$$\mathrm{Li}_2(x) = n \sum_{z^n = x} \mathrm{Li}_2(z) \qquad (n = 1, 2, 3 \ldots)$$

Special Values

Proposition

There are exactly 8 values of z for which z and $\mathrm{Li}_2(z)$ can both be given in closed form:

$$\begin{aligned} \mathrm{Li}_2(0) &= 0 \\ \mathrm{Li}_2(1) &= \frac{\pi^2}{6} \\ \mathrm{Li}_2(-1) &= -\frac{\pi^2}{12} \end{aligned}$$

$$\begin{aligned} \mathrm{Li}_2(\frac{1}{2}) &= \frac{\pi^2}{12} - \frac{1}{2}\log^2(2) \\ \mathrm{Li}_2(\phi) &= \frac{\pi^2}{10} - \log^2(\phi^{-1}) \\ \mathrm{Li}_2(-\phi) &= -\frac{\pi^2}{15} - \frac{1}{2}\log^2(\phi^{-1}) \\ \mathrm{Li}_2(\phi^{-1}) &= \frac{\pi^2}{15} - \log^2(\phi^{-1}) \\ \mathrm{Li}_2(-\phi^{-1}) &= -\frac{\pi^2}{10} - \frac{1}{2}\log^2(\phi^{-1}) \end{aligned}$$

where $\phi = \frac{\sqrt{5}-1}{2}$ is the golden ratio.

Five-Term Relation

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If we let the initial values to be $z_0 = x, z_1 = 1 - xy$ (so $z_2 = y$), then we have:

$$\begin{aligned}
z_3 &= \frac{1 - z_2}{z_1} = \frac{1 - y}{1 - xy} \\
z_4 &= \frac{1 - z_3}{z_2} = \frac{1 - x}{1 - xy} \\
z_5 &= \frac{1 - z_4}{z_3} = x \\
z_6 &= \frac{1 - z_5}{z_4} = 1 - xy \\
&\cdots
\end{aligned}$$

so this recurrence relation actually has period 5!

Five-Term Relation

The most important functional equation is the following:

Theorem (Spence(1809), Abel(1827), Hill(1828), Kummer(1840),Schaeffer(1846)...)

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1 - xy) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right)$$

$$= \frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)(1-y) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right)$$

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The right hand side is a junk — they can be removed by giving an equivalent but modified definition of the dilogarithm function.

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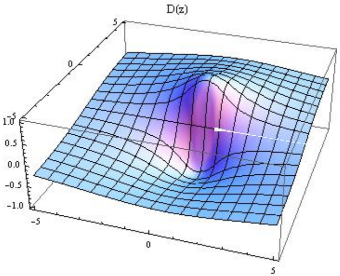

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- (Kummer)

$$D(z) = \frac{1}{2}\left[D(\frac{z}{\bar{z}}) + D(\frac{1-1/z}{1-1/\bar{z}}) + D(\frac{1/(1-z)}{1/(1-\bar{z})})\right]$$

i.e. $D(z)$ only depends on its value on the unit circle:

$$D(e^{i\theta}) = \Im[\mathrm{Li}_2(e^{i\theta})] = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$$

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- (5-term relation)

$$D(x) + D(1-xy) + D(y) + D(\frac{1-y}{1-xy}) + D(\frac{1-x}{1-xy}) = 0$$

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- "6-fold symmetry" says that $\widetilde{D}$ is (anti)invariant under (odd)even permutation of its 4 variabes,
- "5-term relation" becomes

$$\sum_{i=0}^{4}(-1)^i \widetilde{D}(z_0, \ldots, \hat{z}_i, \ldots, z_4) = 0 \qquad (z_0, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C}))$$

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All applications onward will be related to THE 5-term relation.

Ideal Tetrahedra

Let's realize the hyperbolic 3-space as $\mathfrak{H}_3 = \mathbb{C} \times \mathbb{R}_+$ with standard hyperbolic metric.

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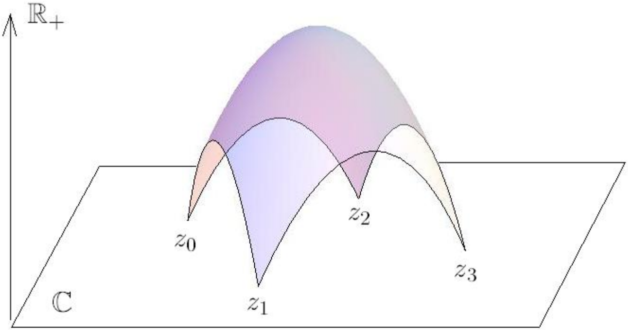
Definition

An *ideal tetrahedron* is a tetrahedron whose vertices are all in $\partial\mathfrak{H}_3 = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$

How does Ideal Tetrahedra look like?



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- "6-fold symmetry" follows from the fact that renumbering the vertices leaves Δ unchanged but may change the orientation.
- "5-term relation" follows from the fact that the five Δ's spanned by 4 at a time of $z_0, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C})$, with signs, add up algebraically to a zero 3-cycle.

Volume of Hyperbolic 3-manifold

- It turns out that the group $SL_2(\mathbb{C})$ acts on $\mathfrak{H}_3$ by isometries, and it can always bring $\{z_0, z_1, z_2, z_3\}$ into the form $\{\infty, 0, 1, z\}$.

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Theorem (Jørgensen and Thurston)

The "volume spectrum"

$$\boldsymbol{Vol} = \{\, Vol(M) | M \text{ a hyperbolic 3-manifold}\} \subset \mathbb{R}_+$$

is a countable and well-ordered subset of $\mathbb{R}_+$.

Bloch Group

- The parameters z_v of the tetrahedra triangulation need to satisfy

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in the abelian group $\bigwedge^2 \mathbb{C}^\times$.
- Here $\bigwedge^2 \mathbb{C}^\times$ is the set of all formal linear combinations $x \wedge y, x, y \in \mathbb{C}^\times$ subject to the relations
$$x \wedge x = 0$$
and
$$(x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y.$$

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- Then it contains the elements:

$$[x] + [\frac{1}{x}], \qquad [x] + [1 - x],$$

$$[x] + [1 - xy] + [y] + [\frac{1-y}{1-xy}] + [\frac{1-x}{1-xy}] \quad (*)$$

corresponding to the symmetries and 5-term relation for $D(z)$.

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The *Bloch Group* $\mathcal{B}_{\mathbb{C}}$ is defined as the quotient of this abelian group with the subgroup generated by the elements $(*)$

Bloch Group

It follows that D extends to a linear map

$$D : \mathcal{B}_{\mathbb{C}} \longrightarrow \mathbb{R}$$

by

$$[z_1] + \ldots + [z_n] \mapsto D(z_1) + \ldots + D(z_n)$$

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In particular, **Vol** *is countable.*

Dedekind Zeta Function

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The *Dedekind Zeta Function* of a number field F is defined as

$$\zeta_F(s) = \prod_{\mathfrak{p} \subset \mathcal{O}_F} \left(1 - \frac{1}{(N\mathfrak{p})^s}\right)^{-1} = \sum_{\mathfrak{a} \subset \mathcal{O}_F} \frac{1}{(N\mathfrak{a})^s}$$

where $\mathcal{O}_F$ is the number ring of F, and $N\mathfrak{a} = |\mathcal{O}_F/\mathfrak{a}|$ is the *norm* of $\mathfrak{a}$.

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When $F = \mathbb{Q}$, this is just the Riemann Zeta function:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \in \mathbb{Z}} \frac{1}{n^s}$$

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is the L-series.
- Here $\left(\frac{d}{n}\right)$ is the Kronecker Symbol, taking values ± 1 or 0 and periodic with period $|d|$ in n.

Examples

For $a = -7$, we have:

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(s) = \left(\sum_{n=1}^{\infty} n^{-s}\right)\left(\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)n^{-s}\right)$$

$$\left(\frac{-7}{n}\right) = \begin{cases} +1 & n \equiv 1, 2, 4 \pmod 7 \\ -1 & n \equiv 3, 5, 6 \pmod 7 \\ 0 & n \equiv 0 \pmod 7 \end{cases}$$

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- It is well known that $\zeta_F(1)$ can be expressed using the usual logarithm (through a term called *regulator*).
- We expect that $\zeta_F(2)$ can be expressed using dilogarithm also.

Examples

- $$\zeta_F(2) = \zeta(2)L(2) = \frac{\pi^2}{6}\sum_{n\geq 1}\left(\frac{d}{n}\right)n^{-2}$$

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$$\zeta_F(2) = \frac{\pi^2}{6\sqrt{|d|}}\sum_{k=1}^{|d|-1}\left(\frac{d}{k}\right)D(e^{2\pi ik/|d|})$$

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- For example:

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{\pi^2}{3\sqrt{7}}(D(e^{2\pi i/7}) + D(e^{4\pi i/7}) - D(e^{6\pi i/7}))$$

expressing $\zeta_F(2)$ in closed form using $D(z)$ at algebraic arguments z.

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- For example:

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{4\pi^2}{21\sqrt{7}} \left(2D(\frac{1+\sqrt{-7}}{2}) + D(\frac{-1+\sqrt{-7}}{4}) \right)$$

Algebraic K Theory

For general number field F with r_1 real and r_2 pairs of complex embeddings, the relation between $D(z)$ and $\zeta_F(2)$ is given nicely through the use of Algebraic K Theory by A. Borel.

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- $\mathcal{B}_F/\{torsion\} \simeq \mathbb{Z}^{r_2}$
- *The image of the map $\mathcal{B}_F \longrightarrow \mathcal{B}_{\mathbb{C}}^{r_2} \xrightarrow{D} \mathbb{R}^{r_2}$ after torsion, has co-volume:*

$$c|d|^{1/2}\zeta_F(2)/\pi^{2r_1+2r_2} \text{ for some } c \in \mathbb{Q}$$

Here the first map corresponds to the r_2 different complex embeddings of F to $\mathbb{C}$.

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Conjecture (Lichtenbaum)

The rational multiple c is related to

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- There exists efficient algorithm to produce many elements in $\mathcal{B}_F$, and $|K_3(F)_{\text{torsion}}|$ is also easy (not for me...) to determine.
- So the mysterious dilogarithm gives, at least conjecturally, an effective way of calculating the orders of certain groups in algebraic K-theory!

Rogers dilogarithm $L(z)$

Another version of dilogarithm, taking real arguments, is more common in physical literature:

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Definition

The Rogers dilogarithm is defined as

$$L(x) := \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x)$$

$$= -\frac{1}{2}\int_0^x \left(\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right) dy$$

and has an analytic continuation to $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$.

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and has an analytic continuation to $\mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$. Furthermore, $L(x)$ belongs to the class $C^\infty((0,1))$.

Rogers dilogarithm

The five-term relation is now simplified to:

$$L(x) + L(1-xy) + L(y) + L(\frac{1-y}{1-xy}) + L(\frac{1-x}{1-xy}) = \frac{\pi^2}{2}$$

where $0 < x, y < 1$.

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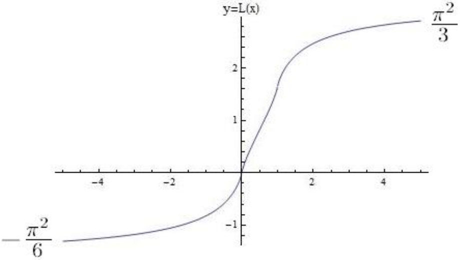
$L(x)$ is extended to the rest of $\mathbb{R}$ by setting $L(0) = 0, L(1) = \frac{\pi^2}{6}$,

$$L(x) = \begin{cases} 2L(1) - L(1/x) & \text{if } x > 1 \\ -L(x/(x-1)) & \text{if } x < 0 \end{cases}$$

Rogers dilogarithm



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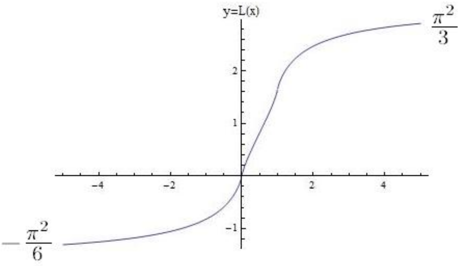

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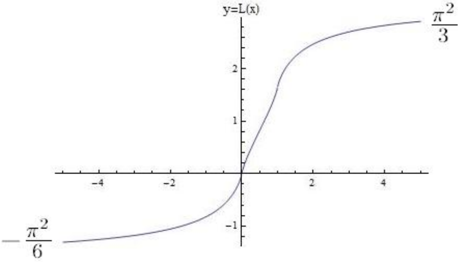

- Modulo $\frac{\pi^2}{2}$, this function is "continuous" at ∞, and the five-term relation holds.
- Every "nice" functional equations is again a consequence of the five-term relation.

Conformal Field Theory

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- For example, consider the identity:

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- The right hand side is related to the effective central charge of the $SU(2)$ level k WZW model,
- on the left hand side, the expression (Jones indices): $\frac{\sin \frac{(i+1)\pi}{k+2}}{\sin \frac{\pi}{k+2}}$ is the "quantum dimensions" of the primary fields of this WZW theory.

Modular Function

- a q Hypergeometric series is (roughly) a series of the form $\sum_{n=0}^{\infty} A_n(q)$ where $\frac{A_n(q)}{A_{n-1}(q)}$ is rational function of q

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- Consider the r-fold q-hypergeometric series defined by:

$$f_{A,B,C}(z) = \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^t A n + B^t n + C}}{(q)_{n_1} \dots (q)_{n_r}}$$

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- This conjecture is motivated from physics: all modular functions $f_{A,B,C}$ obtained in this way should be characters of rational conformal field theories.
- In some special cases, the proof uses Quantum Dilogarithm.

Quantum Dilogarithm

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- The second expression says $S_q(w)$ is like an Exponential function
- The last expression says $S_q(w)$ is like a Dilogarithm function!

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- The last relation reduced to the 5-term relation in a suitable $q \longrightarrow 1$ limit.

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- Using this fact, Hikami constructed a Knot Invariant related to the complement of the hyperbolic volume of links.

Quantum Group

More importantly, the Quantum Dilogarithm is used to construct the Universal R-matrix of $SL_q(2)$, the most important component of Quantum Group Theory:

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Theorem (Drinfeld)

For the quantum group $SL_q(2) = \langle K = q^H, K', e, f\rangle$, the Universal R Matrix is given by:

$$R = q^{-\frac{H\otimes H'}{2}} S_q(-(q-q^{-1})^2 e\otimes f) \in \mathcal{U}_q \otimes \mathcal{U}_q$$

satisfying the Yang-Baxter Relation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Many Other Applications...

- **Combinatorial formula for characteristic classes**
  (Gel'fand, MacPherson...)
- **Cohomology of** $GL_n(\mathbb{C})$
  (A. Borel, Dupont, Quillen...)
- **Rogers-Ramanujan's type identities, asymptotic behavior of partitions**
  (Ramanujan, Hardy, Littlewood...)
- **Representation Theory of infinite dimensional Lie Algebra**
  (Lepowsky, Kac, Fuchs, E.Frenkel...)
- **Exactly Solvable Models**
  (Baxter, Kirillov, Reshetikhin...)
- **Feynman Integral of Ladder Diagrams**
  (Ussyukina, Davydvchev...)
- and much more...

Conclusion

"The dilogarithm function is the only mathematical function with a sense of humor."

– Don Zagier

Reference

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