Positive Representations of Split Real Quantum Groups

Ivan Chi-Ho IP

Kavli IPMU, University of Tokyo
Yale University

Aug 23, 2012

The XXIXth International Colloquium on Group-Theoretical Methods in Physics
Chern Institute of Mathematics, China

References

This talk is based on the following papers:

I. Frenkel-I. Ip, *Positive representations of split real quantum groups and future perspectives*, arXiv:1111.1033

I. Ip, *Positive representations of split real simply-laced quantum groups*, arXiv:1203.2018

I. Ip, *Positive representations of split real quantum groups of type* B_n, C_n, F_4 *and* G_2, arXiv:1205.2940

Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ it's compact form.
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)

Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ it's compact form.
(e.g. for type A_n, $\mathfrak{g} = SL(n+1, \mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)
Representation theory has nice properties:

Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ it's compact form.
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)
Representation theory has nice properties:

- Finite dimensional representations V_λ parametrized by $\lambda \in P^+ \subset \mathfrak{h}_{\mathbb{R}}^*$ (positive weights),

Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ it's compact form.
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)
Representation theory has nice properties:

- Finite dimensional representations V_λ parametrized by $\lambda \in P^+ \subset \mathfrak{h}_\mathbb{R}^*$ (positive weights),
- Peter Weyl: $\mathbb{C}[G] \simeq \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$,

Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ it's compact form.
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)
Representation theory has nice properties:

- Finite dimensional representations V_λ parametrized by $\lambda \in P^+ \subset \mathfrak{h}_\mathbb{R}^*$ (positive weights),
- Peter Weyl: $\mathbb{C}[G] \simeq \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$,
- Closed under tensor product: $V_\lambda \otimes V_\mu \simeq \bigoplus_\nu c_{\lambda\mu}^\nu V_\nu$.

Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ it's compact form.
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)
Representation theory has nice properties:

- Finite dimensional representations V_λ parametrized by $\lambda \in P^+ \subset \mathfrak{h}_\mathbb{R}^*$ (positive weights),
- Peter Weyl: $\mathbb{C}[G] \simeq \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$,
- Closed under tensor product: $V_\lambda \otimes V_\mu \simeq \bigoplus_\nu c_{\lambda\mu}^\nu V_\nu$.

These also hold for compact quantum groups $\mathcal{U}_q(\mathfrak{g}_c)$.

Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ it's compact form.
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)
Representation theory has nice properties:

- Finite dimensional representations V_λ parametrized by $\lambda \in P^+ \subset \mathfrak{h}_\mathbb{R}^*$ (positive weights),
- Peter Weyl: $\mathbb{C}[G] \simeq \displaystyle\bigoplus_\lambda V_\lambda \otimes V_\lambda^*$,
- Closed under tensor product: $V_\lambda \otimes V_\mu \simeq \displaystyle\bigoplus_\nu c_{\lambda\mu}^\nu V_\nu$.

These also hold for compact quantum groups $\mathcal{U}_q(\mathfrak{g}_c)$.
Existence of universal R matrix $\Longrightarrow$ Braided Tensor Category.

Motivation

Considering the split real form $\mathfrak{g}_{\mathbb{R}}$
(e.g. for type A_n, $\mathfrak{g} = SL(n+1, \mathbb{C})$ and $\mathfrak{g}_{\mathbb{R}} = SL(n+1, \mathbb{R})$.)

Motivation

Considering the split real form $\mathfrak{g}_{\mathbb{R}}$
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_{\mathbb{R}} = SL(n+1,\mathbb{R})$.)

The situation is not so nice...

Motivation

Considering the split real form $\mathfrak{g}_{\mathbb{R}}$
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_{\mathbb{R}} = SL(n+1,\mathbb{R})$.)

The situation is not so nice...

Example:
$(SL(2,\mathbb{R}))$: $\mathcal{P}_\lambda \otimes \mathcal{P}_\mu \simeq \bigoplus P_\nu \bigoplus$ (discrete series)...

$L^2(SL(2,\mathbb{R})) \simeq \int^{\oplus} P_\lambda \otimes P_\lambda^* d\lambda \bigoplus$ (discrete series)...

Motivation

Considering the split real form $\mathfrak{g}_\mathbb{R}$
(e.g. for type A_n, $\mathfrak{g} = SL(n+1,\mathbb{C})$ and $\mathfrak{g}_\mathbb{R} = SL(n+1,\mathbb{R})$.)

The situation is not so nice...

Example:
$(SL(2,\mathbb{R}))$: $\mathcal{P}_\lambda \otimes \mathcal{P}_\mu \simeq \bigoplus P_\nu \bigoplus$ (discrete series)...

$L^2(SL(2,\mathbb{R})) \simeq \int^\oplus P_\lambda \otimes P_\lambda^* d\lambda \bigoplus$ (discrete series)...

Parametrization: we can restrict to principal series associated to Borel subalgebra $\mathfrak{b}_\mathbb{R}$, parametrized by $\mathbb{R}_+$-span of P^+.
(Minimal principal series)

$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Let us first recall the definition of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$:

$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Let us first recall the definition of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$:

Definition (Drinfel'd-Jimbo)

Let $q = e^{\pi i b^2}, |q| = 1, b^2 \in (0,1) \setminus \mathbb{Q}$.
$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ *is the Hopf-* algebra generated by* E, F, K *such that*

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K^* = K, \qquad E^* = E, \qquad F^* = F$$

$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Let us first recall the definition of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$:

Definition (Drinfel'd-Jimbo)

Let $q = e^{\pi i b^2}, |q| = 1, b^2 \in (0,1) \setminus \mathbb{Q}$.
$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ *is the Hopf-* * *algebra generated by* E, F, K *such that*

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K^* = K, \qquad E^* = E, \qquad F^* = F$$

Coproduct: $\Delta(K) = K \otimes K$

$$\Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Let us first recall the definition of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$:

Definition (Drinfel'd-Jimbo)

Let $q = e^{\pi i b^2}, |q| = 1, b^2 \in (0,1) \setminus \mathbb{Q}$.
$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ *is the Hopf-* algebra generated by* E, F, K *such that*

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K^* = K, \qquad E^* = E, \qquad F^* = F$$

Coproduct: $\Delta(K) = K \otimes K$

$$\Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

For higher rank, also $E_i F_j = q^{a_{ij}} F_j E_i$, Serre relations etc.

Faddeev's Modular Double

For $|q| = 1$, there is a canonical representation for the relation

$$uv = q^2 vu$$

where u, v are positive self adjoint operators:

Faddeev's Modular Double

For $|q| = 1$, there is a canonical representation for the relation

$$uv = q^2 vu$$

where u, v are positive self adjoint operators:

$$u = e^{2\pi bx}, \qquad v = e^{2\pi bp}$$

unbounded operators on $L^2(\mathbb{R})$. $(p = \frac{1}{2\pi i}\frac{d}{dx})$
(acting densely on the core $\mathcal{W} = span\{e^{-\alpha x^2+\beta x}P(x)\}$.)

Faddeev's Modular Double

For $|q| = 1$, there is a canonical representation for the relation

$$uv = q^2 vu$$

where u, v are positive self adjoint operators:

$$u = e^{2\pi bx}, \qquad v = e^{2\pi bp}$$

unbounded operators on $L^2(\mathbb{R})$. $(p = \frac{1}{2\pi i}\frac{d}{dx})$
(acting densely on the core $\mathcal{W} = span\{e^{-\alpha x^2+\beta x}P(x)\}$.)
Algebraically irreducible, but does not generate full $\mathcal{B}(L^2(\mathbb{R}))$:

Faddeev's Modular Double

For $|q| = 1$, there is a canonical representation for the relation

$$uv = q^2 vu$$

where u, v are positive self adjoint operators:

$$u = e^{2\pi bx}, \qquad v = e^{2\pi bp}$$

unbounded operators on $L^2(\mathbb{R})$. ($p = \frac{1}{2\pi i}\frac{d}{dx}$)
(acting densely on the core $\mathcal{W} = span\{e^{-\alpha x^2+\beta x}P(x)\}$.)
Algebraically irreducible, but does not generate full $\mathcal{B}(L^2(\mathbb{R}))$:
Define $\widetilde{u} := u^{\frac{1}{b^2}}, \widetilde{v} := v^{\frac{1}{b^2}}, \widetilde{q} = e^{\pi i b^{-2}}$ (replacing b by b^{-1}).

Faddeev's Modular Double

For $|q| = 1$, there is a canonical representation for the relation

$$uv = q^2 vu$$

where u, v are positive self adjoint operators:

$$u = e^{2\pi bx}, \qquad v = e^{2\pi bp}$$

unbounded operators on $L^2(\mathbb{R})$. $(p = \frac{1}{2\pi i}\frac{d}{dx})$
(acting densely on the core $\mathcal{W} = span\{e^{-\alpha x^2+\beta x}P(x)\}$.)
Algebraically irreducible, but does not generate full $\mathcal{B}(L^2(\mathbb{R}))$:
Define $\widetilde{u} := u^{\frac{1}{b^2}}, \widetilde{v} := v^{\frac{1}{b^2}}, \widetilde{q} = e^{\pi i b^{-2}}$ (replacing b by b^{-1}).
Then

- $\widetilde{u}\widetilde{v} = \widetilde{q}^2\widetilde{v}\widetilde{u}$

Faddeev's Modular Double

For $|q| = 1$, there is a canonical representation for the relation

$$uv = q^2 vu$$

where u, v are positive self adjoint operators:

$$u = e^{2\pi bx}, \qquad v = e^{2\pi bp}$$

unbounded operators on $L^2(\mathbb{R})$. ($p = \frac{1}{2\pi i}\frac{d}{dx}$)
(acting densely on the core $\mathcal{W} = span\{e^{-\alpha x^2+\beta x}P(x)\}$.)
Algebraically irreducible, but does not generate full $\mathcal{B}(L^2(\mathbb{R}))$:
Define $\widetilde{u} := u^{\frac{1}{b^2}}, \widetilde{v} := v^{\frac{1}{b^2}}, \widetilde{q} = e^{\pi i b^{-2}}$ (replacing b by b^{-1}).
Then

- $\widetilde{u}\widetilde{v} = \widetilde{q}^2\widetilde{v}\widetilde{u}$
- (u, v) commute (weakly) with $(\widetilde{u}, \widetilde{v})$

Faddeev's Modular Double

For $|q|=1$, there is a canonical representation for the relation

$$uv = q^2 vu$$

where u, v are positive self adjoint operators:

$$u = e^{2\pi bx}, \qquad v = e^{2\pi bp}$$

unbounded operators on $L^2(\mathbb{R})$. ($p = \frac{1}{2\pi i}\frac{d}{dx}$)
(acting densely on the core $\mathcal{W} = span\{e^{-\alpha x^2+\beta x}P(x)\}$.)
Algebraically irreducible, but does not generate full $\mathcal{B}(L^2(\mathbb{R}))$:
Define $\widetilde{u} := u^{\frac{1}{b^2}}, \widetilde{v} := v^{\frac{1}{b^2}}, \widetilde{q} = e^{\pi i b^{-2}}$ (replacing b by b^{-1}).
Then

- $\widetilde{u}\widetilde{v} = \widetilde{q}^2\widetilde{v}\widetilde{u}$
- (u, v) commute (weakly) with $(\widetilde{u}, \widetilde{v})$

Together they form the Modular Double.

Ponsot-Teschner's representation

Starting point: Ponsot-Teschner (2003) (Liouville Theory):
Special class of representation $\mathcal{P}_\lambda$ for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ $(\lambda \in \mathbb{R}_+)$:

Ponsot-Teschner's representation

Starting point: Ponsot-Teschner (2003) (Liouville Theory):
Special class of representation $\mathcal{P}_\lambda$ for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ ($\lambda \in \mathbb{R}_+$):

Theorem

The following gives a representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$*:*

$$\begin{aligned} E &= \left(\frac{i}{q-q^{-1}}\right)\left(e^{\pi b(x-\lambda-2p)} + e^{\pi b(-x+\lambda-2p)}\right) \\ F &= \left(\frac{i}{q-q^{-1}}\right)\left(e^{\pi b(x+\lambda+2p)} + e^{\pi b(-x-\lambda+2p)}\right) \\ K &= e^{-2\pi b x} \end{aligned}$$

(Note: $\frac{i}{q-q^{-1}} = (2\sin \pi b^2)^{-1} > 0$.)

Ponsot-Teschner's representation

Starting point: Ponsot-Teschner (2003) (Liouville Theory):
Special class of representation $\mathcal{P}_\lambda$ for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ $(\lambda \in \mathbb{R}_+)$:

Theorem

The following gives a representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$*:*

$$\begin{aligned} E &= \left(\frac{i}{q-q^{-1}}\right)(e^{\pi b(x-\lambda-2p)} + e^{\pi b(-x+\lambda-2p)}) \\ F &= \left(\frac{i}{q-q^{-1}}\right)(e^{\pi b(x+\lambda+2p)} + e^{\pi b(-x-\lambda+2p)}) \\ K &= e^{-2\pi b x} \end{aligned}$$

(Note: $\frac{i}{q-q^{-1}} = (2\sin \pi b^2)^{-1} > 0$.)

$\mathcal{P}_\lambda$ has NO classical limit as $b \longrightarrow 0$!

Ponsot-Teschner's representation

(1) $\mathcal{P}_\lambda$ is parametrized by $\lambda \in \mathbb{R}_+$.

Ponsot-Teschner's representation

(1) $\mathcal{P}_\lambda$ is parametrized by $\lambda \in \mathbb{R}_+$.

(2) Represented by positive (essentially) self-adjoint operators on $L^2(\mathbb{R})$.

Ponsot-Teschner's representation

(1) $\mathcal{P}_\lambda$ is parametrized by $\lambda \in \mathbb{R}_+$.

(2) Represented by positive (essentially) self-adjoint operators on $L^2(\mathbb{R})$.

(3) Replacing b by b^{-1}, gives $\widetilde{E}, \widetilde{F}, \widetilde{K}$ commuting with E, F, K, also a representation of $\mathcal{U}_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$.

Ponsot-Teschner's representation

(1) $\mathcal{P}_\lambda$ is parametrized by $\lambda \in \mathbb{R}_+$.

(2) Represented by positive (essentially) self-adjoint operators on $L^2(\mathbb{R})$.

(3) Replacing b by b^{-1}, gives $\widetilde{E}, \widetilde{F}, \widetilde{K}$ commuting with E, F, K, also a representation of $\mathcal{U}_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$.

- Define

$$e = \left(\frac{i}{q-q^{-1}}\right)^{-1} E, \qquad f = \left(\frac{i}{q-q^{-1}}\right)^{-1} F,$$

we have

$$e^{\frac{1}{b^2}} = \widetilde{e}, \qquad f^{\frac{1}{b^2}} = \widetilde{f}, \qquad K^{\frac{1}{b^2}} = \widetilde{K},$$

called the "transcendental relations".

Properties

Existence of universal R matrix:

$$R = q^{\frac{H\otimes H}{4}} g_b(e \otimes f) q^{\frac{H\otimes H}{4}}$$

Properties

Existence of universal R matrix:

$$R = q^{\frac{H\otimes H}{4}} g_b(e\otimes f) q^{\frac{H\otimes H}{4}}$$

R is invariant under the change $b \longleftrightarrow b^{-1}$.

Properties

Existence of universal R matrix:

$$R = q^{\frac{H\otimes H}{4}} g_b(e\otimes f) q^{\frac{H\otimes H}{4}}$$

R is invariant under the change $b \longleftrightarrow b^{-1}$.

- Here $g_b(x)$ is called quantum dilogarithm.

Properties

Existence of universal R matrix:

$$R = q^{\frac{H\otimes H}{4}} g_b(e\otimes f) q^{\frac{H\otimes H}{4}}$$

R is invariant under the change $b \longleftrightarrow b^{-1}$.

- Here $g_b(x)$ is called quantum dilogarithm.
- $g_b(x)$ is the non-compact analogue of the q-exponential function

Properties

Existence of universal R matrix:

$$R = q^{\frac{H\otimes H}{4}} g_b(e\otimes f) q^{\frac{H\otimes H}{4}}$$

R is invariant under the change $b \longleftrightarrow b^{-1}$.

- Here $g_b(x)$ is called quantum dilogarithm.
- $g_b(x)$ is the non-compact analogue of the q-exponential function
- $|g_b(x)| = 1$ when $x \in \mathbb{R}_{>0}$.

Properties

Closure under tensor product (in the continuous sense):

Theorem (Ponsot-Teschner (2000))

We have

$$\mathcal{P}_\alpha \otimes \mathcal{P}_\beta \simeq \int_{\mathbb{R}_+}^{\oplus} P_\gamma d\mu(\gamma)$$

where $d\mu(\gamma)$ is expressed in terms of (a variant of) quantum dilogarithm.

Properties

Peter-Weyl type Theorem (proposed by Ponsot-Teschner):

Theorem (Ip (2011))

We have

$$L^2(SL_q^+(2,\mathbb{R})) \simeq \int_{\mathbb{R}_+}^{\oplus} P_\alpha \otimes P_\alpha d\mu(\alpha)$$

as a $\mathcal{U}_{q,L} \otimes \mathcal{U}_{q,R}$ *representation.*

Properties

Peter-Weyl type Theorem (proposed by Ponsot-Teschner):

Theorem (Ip (2011))

We have

$$L^2(SL_q^+(2,\mathbb{R})) \simeq \int_{\mathbb{R}_+}^{\oplus} P_\alpha \otimes P_\alpha d\mu(\alpha)$$

as a $\mathcal{U}_{q,L} \otimes \mathcal{U}_{q,R}$ representation.

Here $L^2(SL_q^+(2,\mathbb{R}))$ is a Hilbert space constructed from the GNS representation of a C^*-algebraic version of $SL_q^+(2,\mathbb{R})$.

Properties

Peter-Weyl type Theorem (proposed by Ponsot-Teschner):

Theorem (Ip (2011))

We have

$$L^2(SL_q^+(2,\mathbb{R})) \simeq \int_{\mathbb{R}_+}^{\oplus} P_\alpha \otimes P_\alpha d\mu(\alpha)$$

as a $\mathcal{U}_{q,L} \otimes \mathcal{U}_{q,R}$ *representation.*

Here $L^2(SL_q^+(2,\mathbb{R}))$ is a Hilbert space constructed from the GNS representation of a C^*-algebraic version of $SL_q^+(2,\mathbb{R})$.

The action of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is obtained by dualizing the regular corepresentation of $SL_q^+(2,\mathbb{R})$.

Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_{\mathbb{R}}$?

Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_{\mathbb{R}}$? Namely, we want to construct a class of representation for $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ for $|q| = 1$ such that

Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_{\mathbb{R}}$? Namely, we want to construct a class of representation for $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ for $|q| = 1$ such that

(1) The class of representation is parametrized by $\mathbb{R}_+$-span of P^+, or equivalently, $(\mathbb{R}_+)^{rank(\mathfrak{g})}$

Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_\mathbb{R}$? Namely, we want to construct a class of representation for $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ for $|q|=1$ such that

(1) The class of representation is parametrized by $\mathbb{R}_+$-span of P^+, or equivalently, $(\mathbb{R}_+)^{rank(\mathfrak{g})}$

(2) The action of the generators E_i, F_i, K_i are represented by positive (essentially) self adjoint operators.

Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_{\mathbb{R}}$? Namely, we want to construct a class of representation for $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ for $|q|=1$ such that

(1) The class of representation is parametrized by $\mathbb{R}_+$-span of P^+, or equivalently, $(\mathbb{R}_+)^{rank(\mathfrak{g})}$

(2) The action of the generators E_i, F_i, K_i are represented by positive (essentially) self adjoint operators.

(3) Transcendental relations $\widetilde{X} = X^{\frac{1}{b^2}}$ exist, relating $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ to $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$ (Modular Double).

Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_{\mathbb{R}}$? Namely, we want to construct a class of representation for $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ for $|q| = 1$ such that

(1) The class of representation is parametrized by $\mathbb{R}_+$-span of P^+, or equivalently, $(\mathbb{R}_+)^{rank(\mathfrak{g})}$

(2) The action of the generators E_i, F_i, K_i are represented by positive (essentially) self adjoint operators.

(3) Transcendental relations $\widetilde{X} = X^{\frac{1}{b^2}}$ exist, relating $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ to $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$ (Modular Double).

We call these "positive principal series representations", or "positive representations" for short.

Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_{\mathbb{R}}$?
Namely, we want to construct a class of representation for $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ for $|q| = 1$ such that

(1) The class of representation is parametrized by $\mathbb{R}_+$-span of P^+, or equivalently, $(\mathbb{R}_+)^{rank(\mathfrak{g})}$

(2) The action of the generators E_i, F_i, K_i are represented by positive (essentially) self adjoint operators.

(3) Transcendental relations $\widetilde{X} = X^{\frac{1}{b^2}}$ exist, relating $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ to $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$ (Modular Double).

We call these "positive principal series representations", or "positive representations" for short.
The answer is YES, and have been constructed for all types of $\mathfrak{g}_{\mathbb{R}}$.

Construction

I will use $\mathcal{U}_q(\mathfrak{sl}(3,\mathbb{R}))$ as a toy model. The general idea is similar.

Construction

I will use $\mathcal{U}_q(\mathfrak{sl}(3,\mathbb{R}))$ as a toy model. The general idea is similar.

Theorem (Lusztig Data for total positivity)

Fix a longest element $w_0 = s_{i_1}...s_{i_m} \in W$ in the Weyl group with reduced expression.

Construction

I will use $\mathcal{U}_q(\mathfrak{sl}(3,\mathbb{R}))$ as a toy model. The general idea is similar.

Theorem (Lusztig Data for total positivity)

Fix a longest element $w_0 = s_{i_1}...s_{i_m} \in W$ in the Weyl group with reduced expression. Then the totally positive upper unipotent subgroup $U_{>0}^+$ is parametrized by

$$\begin{array}{ccc} \mathbb{R}_{>0}^m & \longrightarrow & U_{>0}^+ \\ (a_1, ..., a_m) & \mapsto & x_{i_1}(a_1)...x_{i_m}(a_m) \end{array}$$

Construction

I will use $\mathcal{U}_q(\mathfrak{sl}(3,\mathbb{R}))$ as a toy model. The general idea is similar.

Theorem (Lusztig Data for total positivity)

Fix a longest element $w_0 = s_{i_1}...s_{i_m} \in W$ in the Weyl group with reduced expression. Then the totally positive upper unipotent subgroup $U_{>0}^+$ is parametrized by

$$\begin{array}{ccc} \mathbb{R}_{>0}^m & \longrightarrow & U_{>0}^+ \\ (a_1,...,a_m) & \mapsto & x_{i_1}(a_1)...x_{i_m}(a_m) \end{array}$$

where (x_i, χ_i, y_i) is the root subgroup for each root i.

Construction

I will use $\mathcal{U}_q(\mathfrak{sl}(3,\mathbb{R}))$ as a toy model. The general idea is similar.

Theorem (Lusztig Data for total positivity)

Fix a longest element $w_0 = s_{i_1}...s_{i_m} \in W$ in the Weyl group with reduced expression. Then the totally positive upper unipotent subgroup $U_{>0}^+$ is parametrized by

$$\begin{array}{ccc} \mathbb{R}_{>0}^m & \longrightarrow & U_{>0}^+ \\ (a_1,...,a_m) & \mapsto & x_{i_1}(a_1)...x_{i_m}(a_m) \end{array}$$

where (x_i, χ_i, y_i) is the root subgroup for each root i.

Example: choosing $w_0 = s_2 s_1 s_2$ we have

$$U_{>0}^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & bc \\ 0 & 1 & a+c \\ 0 & 0 & 1 \end{pmatrix}$$

Construction

From this we can apply the regular representation

$$g \cdot f(g_+) = [f(g_+g)]_+ \chi_\lambda(g_+g)$$

to get the action of E, F, H

Construction

From this we can apply the regular representation

$$g \cdot f(g_+) = [f(g_+g)]_+ \chi_\lambda(g_+g)$$

to get the action of E, F, H

Example: $e^{tE_2} : \begin{pmatrix} 1 & b & bc \\ 0 & 1 & a+c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

Construction

From this we can apply the regular representation

$$g \cdot f(g_+) = [f(g_+g)]_+ \chi_\lambda(g_+g)$$

to get the action of E, F, H

Example: $e^{tE_2} : \begin{pmatrix} 1 & b & bc \\ 0 & 1 & a+c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

i.e. action on $\mathbb{C}[U_{>0}^+] : f(a,b,c) \mapsto f(a,b,c+t)$

Construction

From this we can apply the regular representation

$$g \cdot f(g_+) = [f(g_+g)]_+ \chi_\lambda(g_+g)$$

to get the action of E, F, H

Example: $e^{tE_2} : \begin{pmatrix} 1 & b & bc \\ 0 & 1 & a+c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

i.e. action on $\mathbb{C}[U_{>0}^+] : f(a,b,c) \mapsto f(a,b,c+t)$

Action of $E_2 : \frac{\partial}{\partial c} : f \mapsto f_c$

Construction

We have

$$
\begin{aligned}
E_1 : f &\mapsto \frac{c}{b} f_a + f_b - \frac{c}{b} f_c \\
E_2 : f &\mapsto f_c \\
F_1 : f &\mapsto -b^2 f_b + ba f_a + 2\lambda_1 b \\
F_2 : f &\mapsto -a^2 f_a - 2ca f_a + bc f_b - c^2 f_c + 2\lambda_2 (a + c) \\
H_1 : f &\mapsto a f_a - 2b f_b + c f_c + 2\lambda_1 \\
H_2 : f &\mapsto -2a f_a + b_b - 2c_c + 2\lambda_2
\end{aligned}
$$

Acting on $\mathbb{C}[U_{>0}^{+}]$.

Construction

We have

$$\begin{aligned} E_1 : f &\mapsto \frac{c}{b} f_a + f_b - \frac{c}{b} f_c \\ E_2 : f &\mapsto f_c \\ F_1 : f &\mapsto -b^2 f_b + ba f_a + 2\lambda_1 b \\ F_2 : f &\mapsto -a^2 f_a - 2ca f_a + bc f_b - c^2 f_c + 2\lambda_2(a+c) \\ H_1 : f &\mapsto a f_a - 2b f_b + c f_c + 2\lambda_1 \\ H_2 : f &\mapsto -2a f_a + b_b - 2c_c + 2\lambda_2 \end{aligned}$$

Acting on $\mathbb{C}[U^+_{>0}]$.
Different from usual regular representation acting on $\mathbb{C}[U^+]$.

Crucial Step: Mellin Transform

Crucial Step: Mellin Transform

Mellin Transform: unitary map $L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}_+)$:

$$f(u) \mapsto F(x) := \frac{1}{2\pi}\int_{\mathbb{R}} f(u)x^{-\frac{1}{2}+iu}du$$

Crucial Step: Mellin Transform

Formally:

$$f(u) \mapsto F(x) := \int f(u) x^u du$$

Crucial Step: Mellin Transform

Formally:

$$f(u) \mapsto F(x) := \int f(u) x^u du$$

So that:

$$x : f(u) \quad \mapsto \quad f(u-1)$$

Crucial Step: Mellin Transform

Formally:

$$f(u) \mapsto F(x) := \int f(u) x^u du$$

So that:

$$\begin{aligned} x : f(u) &\mapsto f(u-1) \\ \frac{\partial}{\partial x} : f(u) &\mapsto (u+1) f(u+1) \end{aligned}$$

Crucial Step: Mellin Transform

Formally:

$$f(u) \mapsto F(x) := \int f(u) x^u du$$

So that:

$$\begin{aligned} x : f(u) &\mapsto f(u-1) \\ \frac{\partial}{\partial x} : f(u) &\mapsto (u+1) f(u+1) \\ x\frac{\partial}{\partial x} : f(u) &\mapsto u f(u) \end{aligned}$$

Crucial Step: Mellin Transform

Formally:

$$f(u) \mapsto F(x) := \int f(u) x^u du$$

So that:

$$\begin{aligned} x : f(u) &\mapsto f(u-1) \\ \frac{\partial}{\partial x} : f(u) &\mapsto (u+1)f(u+1) \\ x\frac{\partial}{\partial x} : f(u) &\mapsto uf(u) \end{aligned}$$

Differential operators becomes finite difference operators!

Crucial Step: Mellin Transform

We have [(a,b,c)$\mapsto$ (u,v,w) and simplifying notations...]

$$\begin{aligned}
E_1 : f &\mapsto (u+1)f(u+1, v+1, w-1) + (1+v-w)f(v+1) \\
E_2 : f &\mapsto (w+1)f(w+1) \\
F_1 : f &\mapsto (2\lambda_1 + u - v + 1)f(v-1) \\
F_2 : f &\mapsto (2\lambda_2 - u + 1)f(u-1) + (2\lambda_2 - 2u + v - w + 1)f(w-1) \\
H_1 : f &\mapsto (u - 2v + w + 2\lambda_1)f \\
H_2 : f &\mapsto (-2u + v - 2w + 2\lambda_2)f
\end{aligned}$$

Acting on functions with $dim(U^+)$ variables.

Crucial Step 2: Quantization

Crucial Step 2: Quantization

Simply quantizing the weights!!

Crucial Step 2: Quantization

Simply quantizing the weights!!

$$\begin{aligned}
E_1 &: [u+1]_q f(u+1, v+1, w-1) + [1+v-w]_q f(v+1) \\
E_2 &: [w+1]_q f(w+1) \\
F_1 &: [2\lambda_1 + u - v + 1]_q f(v-1) \\
F_2 &: [2\lambda_2 - u + 1]_q f(u-1) + [2\lambda_2 - 2u + v - w + 1]_q f(w-1) \\
K_1 = q^{H_1} &: q^{u-2v+w+2\lambda_1} \\
K_2 = q^{H_2} &: q^{-2u+v-2w+2\lambda_2}
\end{aligned}$$

Crucial Step 2: Quantization

Simply quantizing the weights!!

$$\begin{aligned}
E_1 &: [u+1]_q f(u+1, v+1, w-1) + [1+v-w]_q f(v+1) \\
E_2 &: [w+1]_q f(w+1) \\
F_1 &: [2\lambda_1 + u - v + 1]_q f(v-1) \\
F_2 &: [2\lambda_2 - u + 1]_q f(u-1) + [2\lambda_2 - 2u + v - w + 1]_q f(w-1) \\
K_1 = q^{H_1} &: q^{u-2v+w+2\lambda_1} \\
K_2 = q^{H_2} &: q^{-2u+v-2w+2\lambda_2}
\end{aligned}$$

One can check this is a representation for $\mathcal{U}_q(\mathfrak{g})$, without the real structure yet.

Crucial Step 3: Positivity twist

Crucial Step 3: Positivity twist

To obtain positive representations, the trick is to induce a ”twist” in the quantum weight.

Crucial Step 3: Positivity twist

To obtain positive representations, the trick is to induce a "twist" in the quantum weight.

$$[u+1]_q \mapsto \left[\frac{Q}{2b} - i\frac{u}{b}\right]_q, \qquad Q = b + \frac{1}{b}, \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

(Recall that the original variables belongs to $\mathbb{R}_{>0}$. Now we use the correct Mellin transform, where the variable includes a complex part)

Crucial Step 3: Positivity twist

To obtain positive representations, the trick is to induce a "twist" in the quantum weight.

$$[u+1]_q \mapsto \left[\frac{Q}{2b} - i\frac{u}{b}\right]_q, \qquad Q = b + \frac{1}{b}, \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

(Recall that the original variables belongs to $\mathbb{R}_{>0}$. Now we use the correct Mellin transform, where the variable includes a complex part)

However, no more classical limit as $b \longrightarrow 0$!

$$E_2 : [w+1]_q f(w+1) \longrightarrow \left[\frac{Q}{2b} - i\frac{w}{b}\right]_q e^{-2\pi b p_w}$$

Crucial Step 3: Positivity twist

To obtain positive representations, the trick is to induce a "twist" in the quantum weight.

$$[u+1]_q \mapsto \left[\frac{Q}{2b} - i\frac{u}{b}\right]_q, \qquad Q = b + \frac{1}{b}, \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

(Recall that the original variables belongs to $\mathbb{R}_{>0}$. Now we use the correct Mellin transform, where the variable includes a complex part)

However, no more classical limit as $b \longrightarrow 0$!

$$E_2 : [w+1]_q f(w+1) \longrightarrow \left[\frac{Q}{2b} - i\frac{w}{b}\right]_q e^{-2\pi b p_w}$$

Recall that $q = e^{\pi i b^2}$, this can be rewritten as

$$\left(\frac{i}{q - q^{-1}}\right)\left(e^{\pi b(w - 2p_w)} + e^{-\pi b(w - 2p_w)}\right)$$

which is positive, and can also be shown to be (essentially) self adjoint as required!

Construction

Let us denote simply

$$[u]e(-p) := \left[\frac{Q}{2b} - i\frac{u}{b}\right]_q e^{-2\pi bp}$$

Construction

Let us denote simply

$$[u]e(-p) := \left[\frac{Q}{2b} - i\frac{u}{b}\right]_q e^{-2\pi bp}$$

It is understood that this is positive as long as $[p,u] = \frac{1}{2\pi i}$.

Construction

Let us denote simply

$$[u]e(-p) := \left[\frac{Q}{2b} - i\frac{u}{b}\right]_q e^{-2\pi bp}$$

It is understood that this is positive as long as $[p,u] = \frac{1}{2\pi i}$.
Final result:

$$\begin{aligned}
E_1 &: [u]e(-p_u - p_v + p_w) + [v-w]e(-p_v)\\
E_2 &: [w]e(-p_w)\\
F_1 &: [2\lambda_1 + u - v]e(p_v)\\
F_2 &: [2\lambda_2 - u]e(p_u) + [2\lambda_2 - 2u + v - w]e(p_w)\\
K_1 &: e^{\pi b(u-2v+w-2\lambda_1)}\\
K_2 &: e^{\pi b(-2u+v-2w-2\lambda_2)}
\end{aligned}$$

Acting on $L^2(\mathbb{R}^{\dim U^+})$.

Final Step

So far we have constructed the representation for a particular choice of longest element w_0.

Final Step

So far we have constructed the representation for a particular choice of longest element w_0.

However the representation is indeed canonical:

Final Step

So far we have constructed the representation for a particular choice of longest element w_0.

However the representation is indeed canonical:

Theorem

The transformation of the operators of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ corresponding to the change of words $...s_is_js_i... = ...s_js_is_j...$

$$x_i(u)x_j(v)x_i(w) \longleftrightarrow x_j(u')x_i(v')x_j(w')$$

Final Step

So far we have constructed the representation for a particular choice of longest element w_0.
However the representation is indeed canonical:

Theorem

The transformation of the operators of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ *corresponding to the change of words* $...s_is_js_i... = ...s_js_is_j...$

$$x_i(u)x_j(v)x_i(w) \longleftrightarrow x_j(u')x_i(v')x_j(w')$$

is given by

$$X \mapsto \Phi X \Phi^{-1},$$

Final Step

So far we have constructed the representation for a particular choice of longest element w_0.
However the representation is indeed canonical:

Theorem

The transformation of the operators of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ corresponding to the change of words $...s_is_js_i... = ...s_js_is_j...$

$$x_i(u)x_j(v)x_i(w) \longleftrightarrow x_j(u')x_i(v')x_j(w')$$

is given by

$$X \mapsto \Phi X \Phi^{-1},$$

where

$$\Phi = T \circ g_b(e^{\pi b(2p_w - 2p_u + u - v + w)}) g_b^*(e^{\pi b(2p_w - 2p_u - u + v - w)}),$$

is a unitary transform. Here T is a linear transformation of det=1.

Construction

The action of F_i is essentially the Feigin map, and can be obtained directly from any reduced expression of w_0.

Construction

The action of F_i is essentially the Feigin map, and can be obtained directly from any reduced expression of w_0.

$$F_i = \sum_{k=1}^{n} \left[\sum_{l=k}^{n} \left(\sum_{j}^{m} u_j^{l,m} - 2u_i^l \right) + u_i^k + 2\lambda_i \right] e(p_i^k)$$

To find action of E_i in general:

Construction

The action of F_i is essentially the Feigin map, and can be obtained directly from any reduced expression of w_0.

$$F_i = \sum_{k=1}^{n} \left[\sum_{l=k}^{n} \left(\sum_{j}^{m} u_j^{l,m} - 2u_i^l \right) + u_i^k + 2\lambda_i \right] e(p_i^k)$$

To find action of E_i in general:

- Fix any reduced expression of w_0

Construction

The action of F_i is essentially the Feigin map, and can be obtained directly from any reduced expression of w_0.

$$F_i = \sum_{k=1}^{n}\left[\sum_{l=k}^{n}\left(\sum_{j}^{m} u_j^{l,m} - 2u_i^l\right) + u_i^k + 2\lambda_i\right] e(p_i^k)$$

To find action of E_i in general:

- Fix any reduced expression of w_0
- Find the change of words so that the target index i is to the rightmost.

Construction

The action of F_i is essentially the Feigin map, and can be obtained directly from any reduced expression of w_0.

$$F_i = \sum_{k=1}^{n}\left[\sum_{l=k}^{n}\left(\sum_{j}^{m} u_j^{l,m} - 2u_i^l\right) + u_i^k + 2\lambda_i\right] e(p_i^k)$$

To find action of E_i in general:

- Fix any reduced expression of w_0
- Find the change of words so that the target index i is to the rightmost.
- The action is $[u]e(-p_u)$

Construction

The action of F_i is essentially the Feigin map, and can be obtained directly from any reduced expression of w_0.

$$F_i = \sum_{k=1}^{n}\left[\sum_{l=k}^{n}\left(\sum_{j}^{m} u_j^{l,m} - 2u_i^l\right) + u_i^k + 2\lambda_i\right] e(p_i^k)$$

To find action of E_i in general:

- Fix any reduced expression of w_0
- Find the change of words so that the target index i is to the rightmost.
- The action is $[u]e(-p_u)$
- Carry out the (actually very easy) unitary transformation to obtain the desired expression.

Construction

The action of F_i is essentially the Feigin map, and can be obtained directly from any reduced expression of w_0.

$$F_i = \sum_{k=1}^{n}\left[\sum_{l=k}^{n}\left(\sum_{j}^{m} u_j^{l,m} - 2u_i^l\right) + u_i^k + 2\lambda_i\right] e(p_i^k)$$

To find action of E_i in general:

- Fix any reduced expression of w_0
- Find the change of words so that the target index i is to the rightmost.
- The action is $[u]e(-p_u)$
- Carry out the (actually very easy) unitary transformation to obtain the desired expression.

Positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ of all simply-laced type can be computed this way.

Results

Type A_n: (for the best choice of w_0)

Theorem

The action of E_i, F_i, K_i is given by

$$\begin{aligned}
E_i &= \sum_{k=1}^{n-i+1} [u_{i+k-1}^k - u_{i+k}^k] e\left(\sum_{l=1}^{k} (p_{i+l-1}^{l-1} - p_{i+l-1}^l)\right), \\
F_i &= \sum_{k=1}^{i} \left[u_i^k - \sum_{l=k}^{i} (2u_i^l - u_{i-1}^l - u_{i+1}^{l+1}) - 2\lambda_i\right] e(p_i^k), \\
K_i &= e^{\pi b (\sum_{k=1}^{i} (u_{i-1}^k + u_{i+1}^k - 2u_i^k) + 2\lambda_i)},
\end{aligned}$$

Results

Type D_n: (for the best choice of w_0)

Theorem

For $i = 0$ or 1:

$$\begin{aligned} E_i \quad &= \quad \sum_{k=1}^{n-1} [u_{\overline{k+i-1}}^k - u_2^{2k-1}] e\left(\sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_i^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{1-i}^{l_1} - \sum_{l_2=1}^{2k-2} (-1)^{l_2} p_2^{l_2}\right) \\ &+ \sum_{k=1}^{n-2} [u_2^{2k} - u_{\overline{k+i}}^k] e\left(\sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_i^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{1-i}^{l_1} - \sum_{l_2=1}^{2k} (-1)^{l_2} p_2^{l_2}\right) \end{aligned}$$

and for $i \geq 2$,

$$E_i = \sum_{k=1}^{2n-2i-1} [(-1)^k (u_{i+1}^k - u_i^k)] e\left(\sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_i^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{i+1}^{l_1}\right),$$

where $\overline{k} := k \pmod 2 \in \{0,1\}$, and $s_1(k) := 2\left\lceil \frac{k}{2} \right\rceil - 1, s_2(k) := 2\left\lfloor \frac{k}{2} \right\rfloor$.

Number of terms for the action of E_i

	A_n	D_n	E_6	E_7	E_8
E_0		$2n-3$	9	15	27
E_1	n	$2n-3$	1	11	23
E_2	$n-1$	$2n-5$	11	13	25
E_3	$n-2$	$2n-7$	10	16	28
E_4	$\vdots$	$2n-9$	7	17	29
E_5	$\vdots$	$2n-11$	5	7	19
E_6	$\vdots$	$\vdots$		1	23
E_7	$\vdots$	$\vdots$			1
$\vdots$	$\vdots$	$\vdots$			
E_k	$n-k$	$2n-2k-1$			
$\vdots$	$\vdots$	$\vdots$			
E_{n-1}	2	1			
E_n	1				
$Total:$	$\frac{n(n+1)}{2}$	n^2-2	43	80	175

Fun(?) Facts

Choice of reduced expression for w_0 is important.

Fun(?) Facts

Choice of reduced expression for w_0 is important.

Example: For E_8, the best choice
$w_0 = 4\ 34\ 034\ 230432\ 12340321\ 5432103243054321\ 654320345612345034230123456$
$765432103243546503423012345676543203456123450342301234567$ (obtained by inclusions of Lie algebra) gives at most 29 terms.

$$
\begin{array}{ccccccccccccc}
\circ_1 & - & \circ_2 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - & \circ_6 & - & \circ_7 \\
 & & & & | & & & & & & & & \\
 & & & & \circ_0 & & & & & & & &
\end{array}
$$

Fun(?) Facts

Choice of reduced expression for w_0 is important.

Example: For E_8, the best choice
$w_0 = 4\ 34\ 034\ 230432\ 12340321\ 5432103243054321\ 6543203456123450342301234 56$
$765432103243546503423012345676543203456123450342301234567$ (obtained by inclusions of Lie algebra) gives at most 29 terms.

$$\begin{array}{ccccccccccccc} \circ_1 & - & \circ_2 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - & \circ_6 & - & \circ_7 \\ & & & & | & & & & & & & & \\ & & & & \circ_0 & & & & & & & & \end{array}$$

However, the choice for $w_0 = w' w_{A_n} s_0$ gives over 1,000,000 terms for the action of E_3.

Fun(?) Facts

Choice of reduced expression for w_0 is important.

Example: For E_8, the best choice
$w_0 = 4\ 34\ 034\ 230432\ 12340321\ 5432103243054321\ 6543203456123450342301234567$
$76543210324354650342301234567654320345612345034230 1234567$ (obtained by inclusions of Lie algebra) gives at most 29 terms.

$$
\begin{array}{ccccccccccccc}
\circ_1 & - & \circ_2 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - & \circ_6 & - & \circ_7 \\
 & & & & | & & & & & & & & \\
 & & & & \circ_0 & & & & & & & &
\end{array}
$$

However, the choice for $w_0 = w' w_{A_n} s_0$ gives over 1,000,000 terms for the action of E_3.

But from the previous remarks, they are unitary equivalent representations!

Properties

Extending the Feigin map:

Properties

Extending the Feigin map:

Theorem

Denote by $\mathbb{C}[\mathbb{T}] = \mathbb{C}[\boldsymbol{u}_i^{\pm}, \boldsymbol{v}_i^{\pm}]_{i=1}^{r}$ *the Weyl algebra, where* $\boldsymbol{u}_i \boldsymbol{v}_i = q^2 \boldsymbol{v}_i \boldsymbol{u}_i$*,*

Properties

Extending the Feigin map:

Theorem

Denote by $\mathbb{C}[\mathbb{T}] = \mathbb{C}[\boldsymbol{u}_i^{\pm}, \boldsymbol{v}_i^{\pm}]_{i=1}^r$ *the Weyl algebra, where* $\boldsymbol{u}_i \boldsymbol{v}_i = q^2 \boldsymbol{v}_i \boldsymbol{u}_i$*, we have*

$$\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \longrightarrow \mathbb{C}[\mathbb{T}]$$

such that the image of the generators are polynomials,

Properties

Extending the Feigin map:

Theorem

Denote by $\mathbb{C}[\mathbb{T}] = \mathbb{C}[\boldsymbol{u}_i^{\pm}, \boldsymbol{v}_i^{\pm}]_{i=1}^r$ the Weyl algebra, where $\boldsymbol{u}_i \boldsymbol{v}_i = q^2 \boldsymbol{v}_i \boldsymbol{u}_i$, we have

$$\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \longrightarrow \mathbb{C}[\mathbb{T}]$$

such that the image of the generators are polynomials, and only sums appear (positivity).

Properties

Extending the Feigin map:

Theorem

Denote by $\mathbb{C}[\mathbb{T}] = \mathbb{C}[\boldsymbol{u}_i^{\pm}, \boldsymbol{v}_i^{\pm}]_{i=1}^r$ *the Weyl algebra, where* $\boldsymbol{u}_i\boldsymbol{v}_i = q^2\boldsymbol{v}_i\boldsymbol{u}_i$*, we have*

$$\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \longrightarrow \mathbb{C}[\mathbb{T}]$$

such that the image of the generators are polynomials, and only sums appear (positivity).

We also have the existence of universal R-matrix, essentially replacing $\exp_q$ by g_b in the Reshetikhin model, and showing certain positivity properties. (In preparation)

Properties

Transcendental Relations:

Properties

Transcendental Relations:

Theorem

Define

$$e_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} F_i,$$

Properties

Transcendental Relations:

Theorem

Define

$$e_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} F_i,$$

we have

$$e_i^{\frac{1}{b^2}} = \widetilde{e}_i, \qquad f_i^{\frac{1}{b^2}} = \widetilde{f}_i, \qquad K_i^{\frac{1}{b^2}} = \widetilde{K}_i,$$

where $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i$ *generates* $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$. *(replacing* $b \longleftrightarrow b^{-1}$*)*

Properties

Transcendental Relations:

Theorem

Define

$$e_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} F_i,$$

we have

$$e_i^{\frac{1}{b^2}} = \widetilde{e}_i, \qquad f_i^{\frac{1}{b^2}} = \widetilde{f}_i, \qquad K_i^{\frac{1}{b^2}} = \widetilde{K}_i,$$

where $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i$ *generates* $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$. *(replacing* $b \longleftrightarrow b^{-1}$*)*

Follows from the "magic Lemma" by Yu. Volkov: For $u, v > 0$:

$$uv = q^2 vu \Longrightarrow (u+v)^{\frac{1}{b^2}} = u^{\frac{1}{b^2}} + v^{\frac{1}{b^2}}$$

Properties

Transcendental Relations:

Theorem

Define

$$e_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} F_i,$$

we have

$$e_i^{\frac{1}{b^2}} = \widetilde{e}_i, \qquad f_i^{\frac{1}{b^2}} = \widetilde{f}_i, \qquad K_i^{\frac{1}{b^2}} = \widetilde{K}_i,$$

where $\widetilde{E_i}, \widetilde{F_i}, \widetilde{K_i}$ *generates* $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$. *(replacing* $b \longleftrightarrow b^{-1}$*)*

Follows from the "magic Lemma" by Yu. Volkov: For $u, v > 0$:

$$uv = q^2 vu \Longrightarrow (u+v)^{\frac{1}{b^2}} = u^{\frac{1}{b^2}} + v^{\frac{1}{b^2}}$$

However, $(\widetilde{E_i}, \widetilde{F_i}, \widetilde{K_i})$ commute with (E_i, F_i, K_i) only up to a sign.

Properties

Transcendental Relations:

Theorem

Define

$$e_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q-q^{-1}}\right)^{-1} F_i,$$

we have

$$e_i^{\frac{1}{b^2}} = \widetilde{e}_i, \qquad f_i^{\frac{1}{b^2}} = \widetilde{f}_i, \qquad K_i^{\frac{1}{b^2}} = \widetilde{K}_i,$$

where $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i$ *generates* $\mathcal{U}_{\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$. *(replacing* $b \longleftrightarrow b^{-1}$*)*

Follows from the "magic Lemma" by Yu. Volkov: For $u, v > 0$:

$$uv = q^2 vu \Longrightarrow (u+v)^{\frac{1}{b^2}} = u^{\frac{1}{b^2}} + v^{\frac{1}{b^2}}$$

However, $(\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i)$ commute with (E_i, F_i, K_i) only up to a sign.
$\Longrightarrow$ Need a slight modification in order to define the modular double.

Properties

After modifying the quantum group by some scaling of K_i's, we then have the following result for $\mathbf{U}_q(\mathfrak{g}_\mathbb{R})$:

Properties

After modifying the quantum group by some scaling of K_i's, we then have the following result for $\mathbf{U}_q(\mathfrak{g}_\mathbb{R})$:

Theorem

The commutant of $\boldsymbol{U}_q(\mathfrak{g}_\mathbb{R})$ is the Langlands dual of the modular double counterpart,

$$(\boldsymbol{U}_q(\mathfrak{g}_\mathbb{R}))' = \boldsymbol{U}_{\widetilde{q}}({}^L\mathfrak{g}_\mathbb{R})$$

Non-simply-laced case

Positive representations constructed using similar techniques.
(Different reduced expressions of $w_0 \Longrightarrow$ transformations are more complicated)

Non-simply-laced case

Positive representations constructed using similar techniques.
(Different reduced expressions of $w_0 \Longrightarrow$ transformations are more complicated)
(However, related to simply-laced case by certain folding methods.)

Non-simply-laced case

Positive representations constructed using similar techniques.
(Different reduced expressions of $w_0 \Longrightarrow$ transformations are more complicated)
(However, related to simply-laced case by certain folding methods.)

Surprising discovery using transcendental relations.

Non-simply-laced case

Positive representations constructed using similar techniques.
(Different reduced expressions of $w_0 \Longrightarrow$ transformations are more complicated)
(However, related to simply-laced case by certain folding methods.)

Surprising discovery using transcendental relations.

Transcendental relations no longer exchange $b \longleftrightarrow b^{-1}$.

Non-simply-laced case

Positive representations constructed using similar techniques.
(Different reduced expressions of $w_0 \Longrightarrow$ transformations are more complicated)
(However, related to simply-laced case by certain folding methods.)

Surprising discovery using transcendental relations.

Transcendental relations no longer exchange $b \longleftrightarrow b^{-1}$.

Instead, it exchanges short roots and long roots!!

Non-simply-laced case

Positive representations constructed using similar techniques.
(Different reduced expressions of $w_0 \Longrightarrow$ transformations are more complicated)
(However, related to simply-laced case by certain folding methods.)

Surprising discovery using transcendental relations.

Transcendental relations no longer exchange $b \longleftrightarrow b^{-1}$.

Instead, it exchanges short roots and long roots!!

Non-simply-laced case

Theorem

Let $q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)} = e^{\pi i b_i^2}$. *Define*

$$e_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} F_i,$$

Non-simply-laced case

Theorem

Let $q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)} = e^{\pi i b_i^2}$. *Define*

$$e_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} F_i,$$

and define

$$\widetilde{e_i} := e_i^{\frac{1}{b_i^2}}, \qquad \widetilde{f_i} := f_i^{\frac{1}{b_i^2}}, \qquad \widetilde{K_i} := K_i^{\frac{1}{b_i^2}},$$

Non-simply-laced case

Theorem

Let $q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)} = e^{\pi i b_i^2}$. *Define*

$$e_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} F_i,$$

and define

$$\widetilde{e_i} := e_i^{\frac{1}{b_i^2}}, \qquad \widetilde{f_i} := f_i^{\frac{1}{b_i^2}}, \qquad \widetilde{K_i} := K_i^{\frac{1}{b_i^2}},$$

then $\widetilde{E_i}, \widetilde{F_i}, \widetilde{K_i}$ *generates* $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$.

Non-simply-laced case

Theorem

Let $q_i = q^{\frac{1}{2}(\alpha_i,\alpha_i)} = e^{\pi i b_i^2}$. *Define*

$$e_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} E_i, \qquad f_i = \left(\frac{i}{q_i - q_i^{-1}}\right)^{-1} F_i,$$

and define

$$\widetilde{e_i} := e_i^{\frac{1}{b_i^2}}, \qquad \widetilde{f_i} := f_i^{\frac{1}{b_i^2}}, \qquad \widetilde{K_i} := K_i^{\frac{1}{b_i^2}},$$

then $\widetilde{E_i}, \widetilde{F_i}, \widetilde{K_i}$ *generates* $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$.
If E_i *is the short root in* $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$, *then* $\widetilde{E_i}$ *is the long root in* $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$ *and vice versa.*

Non-simply-laced case

This provides for the first time a very direct analytic relation between $\mathfrak{g}$ and its Langlands dual.

Non-simply-laced case

This provides for the first time a very direct analytic relation between $\mathfrak{g}$ and its Langlands dual.

Note that under this framework, there is still no classical limit.

Future Perspectives

Conjecture

The class of positive representations is closed under tensor product (in the continuous sense), hence form a (certain kind of) Braided Tensor Category.

Future Perspectives

Conjecture

The class of positive representations is closed under tensor product (in the continuous sense), hence form a (certain kind of) Braided Tensor Category.

Indeed using the magic Lemma, we have

$$\Delta(e_i)^{\frac{1}{b^2}} = \Delta\widetilde{e_i}, \qquad \Delta(f_i)^{\frac{1}{b^2}} = \Delta\widetilde{f_i}$$

Future Perspectives

Conjecture

The class of positive representations is closed under tensor product (in the continuous sense), hence form a (certain kind of) Braided Tensor Category.

Indeed using the magic Lemma, we have

$$\Delta(e_i)^{\frac{1}{b^2}} = \Delta\widetilde{e_i}, \qquad \Delta(f_i)^{\frac{1}{b^2}} = \Delta\widetilde{f_i}$$

hence it is enough to show that the defining properties characterize this class.

Future Perspectives

Conjecture

The class of positive representations is closed under tensor product (in the continuous sense), hence form a (certain kind of) Braided Tensor Category.

Indeed using the magic Lemma, we have

$$\Delta(e_i)^{\frac{1}{b^2}} = \Delta\widetilde{e_i}, \qquad \Delta(f_i)^{\frac{1}{b^2}} = \Delta\widetilde{f_i}$$

hence it is enough to show that the defining properties characterize this class.

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT?

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT? $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT? $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.
- Geometrization and Categorification of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$?

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT?
  $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.
- Geometrization and Categorification of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$?
  (H. Nakajima) Gauge theory $\Longrightarrow$ geometric and categorical constructions of fin. dim. reps for $\mathcal{U}_q(\mathfrak{g})$.

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT?
  $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.
- Geometrization and Categorification of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$?
  (H. Nakajima) Gauge theory $\Longrightarrow$ geometric and categorical constructions of fin. dim. reps for $\mathcal{U}_q(\mathfrak{g})$.
  (T. Dimofte,...) Geometric approach based on CSW model for $G_{\mathbb{R}}$ $\Longrightarrow$ relation with the $N = 2$ SUSY gauge theory on a 3-d sphere.

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT?
  $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.
- Geometrization and Categorification of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$?
  (H. Nakajima) Gauge theory $\Longrightarrow$ geometric and categorical constructions of fin. dim. reps for $\mathcal{U}_q(\mathfrak{g})$.
  (T. Dimofte,...) Geometric approach based on CSW model for $G_{\mathbb{R}}$ $\Longrightarrow$ relation with the $N=2$ SUSY gauge theory on a 3-d sphere.
  First step towards geometrization of the category of positive representations of the modular double.

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT?
  $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.
- Geometrization and Categorification of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$?
  (H. Nakajima) Gauge theory $\Longrightarrow$ geometric and categorical constructions of fin. dim. reps for $\mathcal{U}_q(\mathfrak{g})$.
  (T. Dimofte,...) Geometric approach based on CSW model for $G_\mathbb{R}$ $\Longrightarrow$ relation with the $N=2$ SUSY gauge theory on a 3-d sphere.
  First step towards geometrization of the category of positive representations of the modular double.
- Positivity $\Longrightarrow$ new insight into canonical bases, cluster algebra, tropicalization... (Fock-Goncharov, Fomin-Zelevinsky...)

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT? $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.
- Geometrization and Categorification of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$?
  (H. Nakajima) Gauge theory $\Longrightarrow$ geometric and categorical constructions of fin. dim. reps for $\mathcal{U}_q(\mathfrak{g})$.
  (T. Dimofte,...) Geometric approach based on CSW model for $G_{\mathbb{R}}$ $\Longrightarrow$ relation with the $N=2$ SUSY gauge theory on a 3-d sphere.
  First step towards geometrization of the category of positive representations of the modular double.
- Positivity $\Longrightarrow$ new insight into canonical bases, cluster algebra, tropicalization... (Fock-Goncharov, Fomin-Zelevinsky...)
- Semi-group structure and non-existence of classical limit of positive representations $\Longrightarrow$ causality and renormalization.

Future Perspectives

- Braided tensor category structure $\Longrightarrow$ new class of TQFT?
  $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ case: an alternative approach to a new class of TQFT from quantum Teichmüller theory.
- Geometrization and Categorification of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_{\mathbb{R}})$?
  (H. Nakajima) Gauge theory $\Longrightarrow$ geometric and categorical constructions of fin. dim. reps for $\mathcal{U}_q(\mathfrak{g})$.
  (T. Dimofte,...) Geometric approach based on CSW model for $G_{\mathbb{R}}$ $\Longrightarrow$ relation with the $N=2$ SUSY gauge theory on a 3-d sphere.
  First step towards geometrization of the category of positive representations of the modular double.
- Positivity $\Longrightarrow$ new insight into canonical bases, cluster algebra, tropicalization... (Fock-Goncharov, Fomin-Zelevinsky...)
- Semi-group structure and non-existence of classical limit of positive representations $\Longrightarrow$ causality and renormalization.

Thank you!