CHAPTER 1

Abstract Vector Spaces

1.1 Vector Spaces

Let K be a *field*, i.e. a "number system" where you can add, subtract, multiply and divide.
In this course **we will take K to be $\mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$**.

Definition 1.1. A **vector space** over K is a set V together with two operations: $+$ (*addition*) and $\cdot$ (*scalar multiplication*) subject to the following 10 rules for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in K$:

(+1) Closure under addition: $\mathbf{u} \in V, \mathbf{v} \in V \Longrightarrow \mathbf{u} + \mathbf{v} \in V$.

(+2) Addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(+3) Addition is associative: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(+4) Zero exists: there exists $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(+5) Inverse exists: for every $\mathbf{u} \in V$, there exists $\mathbf{u}' \in V$ such that $\mathbf{u} + \mathbf{u}' = \mathbf{0}$. We write $\mathbf{u}' := -\mathbf{u}$.

($\cdot$1) Closure under multiplication: $c \in K, \mathbf{u} \in V \Longrightarrow c \cdot \mathbf{u} \in V$.

($\cdot$2) Multiplication is associative: $(cd) \cdot \mathbf{u} = c \cdot (d \cdot \mathbf{u})$.

($\cdot$3) Multiplication is distributive: $c \cdot (\mathbf{u} + \mathbf{v}) = c \cdot \mathbf{u} + c \cdot \mathbf{v}$.

($\cdot$4) Multiplication is distributive: $(c + d) \cdot \mathbf{u} = c \cdot \mathbf{u} + d \cdot \mathbf{u}$.

($\cdot$5) Unity: $1 \cdot \mathbf{u} = \mathbf{u}$.

The elements of a vector space V are called *vectors*.

Note. We will denote a vector with boldface $\mathbf{u}$ in this note, but you should use $\overrightarrow{u}$ for handwriting. Sometimes we will omit the $\cdot$ for scalar multiplication if it is clear from the context.

Note. Unless otherwise specified, all vector spaces in the examples below is over $\mathbb{R}$.

The following facts follow from the definitions

Properties 1.2. For any $u \in V$ and $c \in K$:

- The zero vector $\mathbf{0} \in V$ is unique.
- The negative vector $-\mathbf{u} \in V$ is unique.
- $0 \cdot \mathbf{u} = \mathbf{0}$.
- $c \cdot \mathbf{0} = \mathbf{0}$.
- $-\mathbf{u} = (-1) \cdot \mathbf{u}$.

Examples of vector spaces over $\mathbb{R}$:

Example 1.1. The space $\mathbb{R}^n$, $n \geq 1$ with the usual vector addition and scalar multiplication.

Example 1.2. $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Example 1.3. The subset $\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0\} \subset \mathbb{R}^3$.

Example 1.4. Real-valued functions $f(t)$ defined on $\mathbb{R}$.

Example 1.5. The set of real-valued differentiable functions satisfying the differential equations

$$f + \frac{d^2 f}{dx^2} = 0.$$

Examples of vector spaces over a field K:

Example 1.6. The zero vector space $\{\mathbf{0}\}$.

Example 1.7. Polynomials with coefficients in K:

$$p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$$

with $a_i \in K$ for all i.

Example 1.8. The set $M_{m \times n}(K)$ of $m \times n$ matrices with entries in K.

Counter-Examples: these are not vector spaces:

Non-Example 1.9. $\mathbb{R}$ is not a vector space over $\mathbb{C}$.

Non-Example 1.10. The first quadrant $\{\begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$.

Non-Example 1.11. The set of all invertible 2×2 matrices.

Non-Example 1.12. Any straight line in $\mathbb{R}^2$ not passing through the origin.

Non-Example 1.13. The set of polynomials of degree exactly n.

Non-Example 1.14. The set of functions satisfying $f(0) = 1$.

1.2 Subspaces

To check whether a subset $H \subset V$ is a vector space, we only need to check zero and closures.

Definition 1.3. A **subspace** of a vector space V is a subset H of V such that

(1) $\mathbf{0} \in H$.

(2) Closure under addition: $\mathbf{u} \in H, \mathbf{v} \in H \implies \mathbf{u} + \mathbf{v} \in H$.

(3) Closure under multiplication: $\mathbf{u} \in H, c \in K \implies c \cdot \mathbf{u} \in H$.

Example 1.15. Every vector space has a zero subspace $\{\mathbf{0}\}$.

Example 1.16. A plane in $\mathbb{R}^3$ through the origin is a subspace of $\mathbb{R}^3$.

Example 1.17. Polynomials of degree at most n with coefficients in K, **written as** $\mathbb{P}_n(K)$, is a subspace of the vector space of all polynomials with coefficients in K.

Example 1.18. Real-valued functions satisfying $f(0) = 0$ is a subspace of the vector space of all real-valued functions.

Non-Example 1.19. Any straight line in $\mathbb{R}^2$ not passing through the origin is not a vector space.

Non-Example 1.20. $\mathbb{R}^2$ is **not** a subspace of $\mathbb{R}^3$. But $\{\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x \in \mathbb{R}, y \in \mathbb{R}\} \subset \mathbb{R}^3$, which looks exactly like $\mathbb{R}^2$, is a subspace.

Definition 1.4. Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_p\}$ be a set of vectors in V. A **linear combination** of S is a any sum of the form

$$c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p \in V, \qquad c_1, ..., c_p \in K.$$

The set **spanned** by S is the set of all *linear combinations* of S, denoted by $Span(S)$.

Remark. More generally, if S is an infinite set, we define

$$Span(S) = \left\{ \sum_{i=1}^{N} c_i\mathbf{v}_i : c_i \in K, \mathbf{v}_i \in S \right\}$$

i.e. the set of all linear combinations which are finite sum. It follows that $Span(V) = V$ if V is a vector space.

Theorem 1.5. $Span(S)$ is a subspace of V.

Theorem 1.6. H is a subspace of V if and only if H is non-empty and closed under linear combinations, i.e.

$$c_i \in K, \mathbf{v}_i \in H \Longrightarrow c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p \in H.$$

Example 1.21. The set $H := \{\begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} \in \mathbb{R}^4 : a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$, since every element of H can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$a\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2 \in \mathbb{R}^4.$$

Hence $H = Span(\mathbf{v}_1, \mathbf{v}_2)$ is a subspace by Theorem 1.6.

1.3 Linearly Independent Sets

Definition 1.7. A set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_p\} \subset V$ is **linearly dependent** if

$$c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

for some $c_i \in K$, not all of them zero.

Linearly independent set are those vectors that are not linearly dependent:

Definition 1.8. A set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_p\} \subset V$ is **linearly independent** if

$$c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

implies $c_i = 0$ for all i.

Example 1.22. A set of one element $\{\mathbf{v}\}$ is linearly independent iff $\mathbf{v} \neq \mathbf{0}$.

Example 1.23. A set of two nonzero element $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent iff $\mathbf{u}$ is not a multiple of $\mathbf{v}$.

Example 1.24. Any set containing $\mathbf{0}$ is linearly dependent.

Example 1.25. The set of vectors $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\}$ is linearly independent.

Example 1.26. The set of polynomials $\{t^2, t, 4t - t^2\}$ is linearly dependent.

Example 1.27. The set of functions $\{\sin t, \cos t\}$ is linearly independent. The set $\{\sin 2t, \sin t \cos t\}$ is linearly dependent.

1.4 Bases

Definition 1.9. Let H be a subspace of a vector space V. A set of vectors $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_m\} \subset V$ is a **basis** for H iff

(1) $\mathcal{B}$ is a linearly independent set.

(2) $H = Span(\mathcal{B})$.

Note. The plural of "basis" is **"bases"**.

Example 1.28. The columns of the $n \times n$ identity matrix I_n:

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, ..., \mathbf{e}_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$$

form the **standard basis** for $\mathbb{R}^n$.

Example 1.29. In general, the columns of an invertible matrix $A \in M_{n\times n}(\mathbb{R})$ form a basis of $\mathbb{R}^n$, because $A\mathbf{x} = \mathbf{0}$ only has trivial solution.

Example 1.30. The polynomials $\{1, t, t^2, ..., t^n\}$ from the **standard basis** for $\mathbb{P}_n(\mathbb{R})$.

Theorem 1.10 (Spanning Set Theorem)**.** Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_p\}$ be a set in V and let $H = Span(S)$.

(1) If one of the vectors, say $\mathbf{v}_k$, is a linear combination of the remaining vectors in S, then $H = Span(S \setminus \{\mathbf{v}_k\})$.

(2) If $H \neq \{\mathbf{0}\}$, some subset of S is a basis of H.

Theorem 1.11 (Unique Representation Theorem)**.** If $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\} \subset V$ is a basis for V, then for each $x \in V$, there exists **unique** scalars $c_1, ..., c_n \in K$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + ... + c_n\mathbf{b}_n.$$

$c_1, ..., c_n$ are called the **coordinates** of $\mathbf{x}$ relative to the basis $\mathcal{B}$, and

$$[x]_{\mathcal{B}} := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$$

is the **coordinate vector** of $\mathbf{x}$ relative to $\mathcal{B}$.

Example 1.31. The coordinate vector of the polynomial $\mathbf{p} = t^3 + 2t^2 + 3t + 4 \in \mathbb{P}_3(\mathbb{R})$ relative to the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ is

$$[\mathbf{p}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^4.$$

We will study the change of basis later.

1.5 Dimensions

Theorem 1.12 (Replacement Theorem)**.** If $V = Span(\mathbf{v}_1, ..., \mathbf{v}_n)$, and $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ is linearly independent set in V, then $m \leq n$.

Proof. (Idea) One can *replace* some $\mathbf{v}_i$ by $\mathbf{u}_1$ so that $\{\mathbf{u}_1, \mathbf{v}_1, ..., \mathbf{v}_n\} \setminus \{\mathbf{v}_i\}$ also spans V. Assume on the contrary that $m > n$. Repeating the process we can replace all $\mathbf{v}$'s by $\mathbf{u}$'s, so that $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ spans V, hence $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ is linearly dependent. □

Applying this statement to different bases $\mathcal{B}$ and $\mathcal{B}'$, which are both spanning and linearly independent, we get

Theorem 1.13. If a vector space V has a basis of n vectors, then every basis of V must also consists of exactly n vectors.

By this Theorem, the following definition makes sense:

Definition 1.14. If V is spanned by a finite set, then V is said to be **finite dimensional**, $dim(V) < \infty$. The **dimension** of V is the number of vectors in any basis $\mathcal{B}$ of V:

$$\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\} \Longrightarrow \dim V := |\mathcal{B}| = n.$$

If $V = \{\mathbf{0}\}$ is the zero vector space, $\dim V := 0$..

If V is not spanned by a finite set, it is **infinite dimensional**, $dim(V) := \infty$.

Note. If the vector space is over the field K we will write $\dim_K V$. If it is over $\mathbb{R}$ or if the field is not specified (as in the Definition above), we simply write $\dim V$ instead.

Example 1.32. $\dim \mathbb{R}^n = n$.

Example 1.33. $\dim_K \mathbb{P}_n(K) = n+1$. The space of all polynomials is infinite dimensional.

Example 1.34. $\dim_K M_{m\times n}(K) = mn$.

Example 1.35. Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Then $\dim V = 2$.

Example 1.36. The space of real-valued functions on $\mathbb{R}$ is infinite dimensional.

Example 1.37. $\dim_{\mathbb{R}} \mathbb{C} = 2$ but $\dim_{\mathbb{C}} \mathbb{C} = 1$. $\dim_{\mathbb{R}} \mathbb{R} = 1$ but $\dim_{\mathbb{Q}} \mathbb{R} = \infty$.

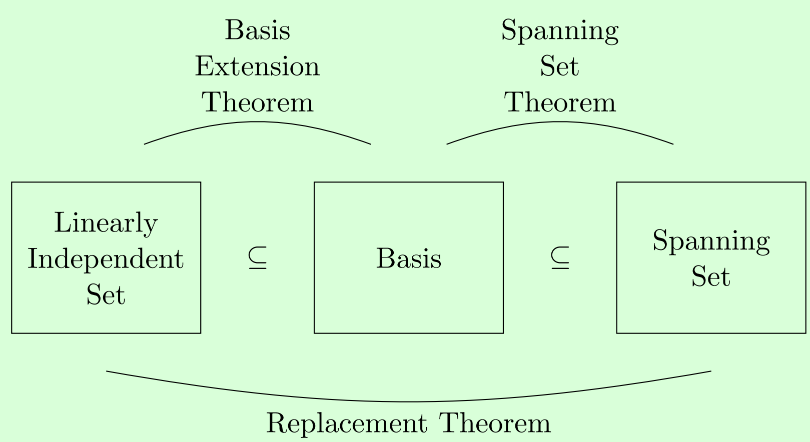
Theorem 1.15 (Basis Extension Theorem)**.** Let H be a subspace of V with $\dim V < \infty$. Any linearly independent set in H can be expanded, if necessary, to a basis for H.
Also, H is finite dimensional and

$$\dim H \leq \dim V.$$

Example 1.38. Subspaces of $\mathbb{R}^3$ are classified as follows:

- 0-dimensional subspaces: only the zero space $\{\mathbf{0}\}$.
- 1-dimensional subspaces: any line passing through origin.
- 2-dimensional subspaces: any plane passing through origin.
- 3-dimensional subspaces: only $\mathbb{R}^3$ itself.

Summarize: For $\dim V < \infty$ and $H \subset V$ a subspace:



If you know the dimension of V, the following Theorem gives a useful criterion to check whether a set is a basis:

Theorem 1.16 (The Basis Theorem)**.** Let V be an n-dimensional vector space, $n \geq 1$, and $S \subset V$ a finite subset with exactly n elements. Then

(1) If S is linearly independent, then S is a basis for V.

(2) If S spans V, then S is a basis for V.

1.6 Intersections, Sums and Direct Sums

We discuss three important construction of vector spaces.

Definition 1.17. Let U, W be subspaces of V.

- $U \cap W$ is the **intersection** of U and W.
- $U + W = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in W\} \subset V$ is the **sum** of U and W.

Properties 1.18.

- $U \cap W$ and $U + W$ are both vector subspaces of V.

Definition 1.19. Let U, W be subspaces of V. Then V is called the **direct sum** of U and W, written as $V = U \oplus W$ if

(1) $V = U + W$.

(2) $U \cap W = \{\mathbf{0}\}$.

Example 1.39. $\mathbb{R}^3 = \{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R}\} \oplus \{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R}\}$.

Example 1.40. $\{\text{Space of functions}\} = \left\{ \begin{array}{c} \text{Even functions} \\ f(-t) = f(t) \end{array} \right\} \oplus \left\{ \begin{array}{c} \text{Odd functions} \\ f(-t) = -f(t) \end{array} \right\}$.

Example 1.41. $\{\text{Matrices}\} = \left\{ \begin{array}{c} \text{Symmetric matrices} \\ \mathbf{A}^T = \mathbf{A} \end{array} \right\} \oplus \left\{ \begin{array}{c} \text{Anti-symmetric matrices} \\ \mathbf{A}^T = -\mathbf{A} \end{array} \right\}$.

Example 1.42. $\{\text{Polynomials}\} = \{\text{Constants}\} \oplus \{\mathbf{p}(t) : \mathbf{p}(0) = 0\}$.

Theorem 1.20. $V = U \oplus W$ iff every $\mathbf{v} \in V$ can be written **uniquely** as

$$\mathbf{v} = \mathbf{u} + \mathbf{w}$$

where $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

Theorem 1.21 (Dimension formula)**.**

$$\dim U + \dim W = \dim(U + W) + \dim(U \cap W).$$

In particular

$$\dim U + \dim W = \dim(U \oplus W).$$

Example 1.43. If U and W are two *different* planes passing through origin in $\mathbb{R}^3$, then $U \cap W$ must be a line and $U + W = \mathbb{R}^3$. The dimension formula then gives $2 + 2 = 3 + 1$.