Introduction to Cluster Algebra

Homework 3 (Due 2017/07/03)

(Q1) Show that matrix mutations preserve the rank of $\widetilde{B}$.

(Hint: Using the matrix mutation rule for $\widetilde{B}$:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [b_{kj}]_+ b_{ik} + [-b_{ik}]_+ b_{kj} & \text{otherwise} \end{cases}$$

Show that one can write

$$\widetilde{B}' = X_m \widetilde{B} Y_n$$

for some $m \times m$ invertible matrix X_m and $n \times n$ invertible matrix Y_n.)

(Q2) Let $\alpha_i \in \mathfrak{h}^*$ be the simple roots. Recall that the corresponding coroots $\alpha_i^\vee \in \mathfrak{h}$ is defined by

$$\alpha_j(\alpha_i^\vee) = c_{ij}.$$

The fundamental weights $\omega_i \in \mathfrak{h}^*$ are defined by

$$\omega_j(\alpha_i^\vee) = \delta_{ij}.$$

(a) Write ω_j in terms of the simple roots α_k.

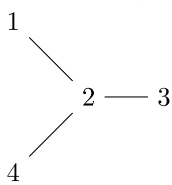
(b) What is the action of the simple reflections $s_i \in W$ on ω_j?

(Q3) Let $G = SL_4$.

(a) Draw the quiver diagram $\Gamma(\mathbf{i})$ corresponding to G^{s_2,w_0}.

(b) What is the cluster type of $\mathbb{C}[G^{s_2,w_0}]$?

(Q4) Let G be of type D_4. Let the reduced word of w_0 be $\mathbf{i} = (1,3,4,2,1,3,4,2,1,3,4,2)$.



(a) Draw the quiver diagram $\Gamma(\mathbf{i})$ corresponding to G^{e,w_0}.

(b) Show that the cluster algebra is of infinite type. (Find a subdiagram of the mutable part of $\Gamma(\mathbf{i})$ that is not 2-finite (see Lecture 6))

(Q5) Let $G = SL_{n+1}$ and $H \subset G$ the maximal torus consisting of the diagonal matrices. Let $h_j(a) := diag(1, ..., \underbrace{a, a^{-1}}_{j,j+1}, ..., 1) \in H$. Then the fundamental weights ω_i act multiplicatively as

$$h_j(a)^{\omega_i} := \omega_i(h_j(a)) = \begin{cases} a & i = j \\ 1 & i \neq j \end{cases} .$$

(a) For $h \in H$, show that $h^{\omega_i} = \prod_{k=1}^{i} h_{kk} = \Delta_{[1,i],[1,i]}(h)$

(b) Hence if $x = [x]_-[x]_0[x]_+ \in N_-HN_+ \subset G$ admits a Gauss decomposition, show that $[x]_0^{\omega_i} = \Delta_{[1,i],[1,i]}(x)$. (Hint: write x as block matrix.)