Lecture Notes
Introduction to Cluster Algebra

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11 Cluster Algebra from Surfaces

In this lecture, we will define and give a quick overview of some properties of cluster algebra from surfaces. We will follow [Surface-I] and [Surface-II].

11.1 Bordered Surfaces with Marked Points

Definition 11.1. *A bordered surface with marked points is a pair* $(\boldsymbol{S}, \boldsymbol{M})$ *where*

- $\boldsymbol{S}$ *a connected oriented 2-dimensional Riemann surface (possibly with boundary* $\partial\boldsymbol{S}$*).*
- $\boldsymbol{M} \subset \boldsymbol{S}$ *a non-empty finite set of* marked points *in* $\boldsymbol{S}$
- $m \in \boldsymbol{M}$ *in the interior* $\boldsymbol{S} - \partial\boldsymbol{S}$ *are called punctures*
- *Each connected boundary component has at least one marked point*
- *We require* $\boldsymbol{S}$ *to have at least one triangulation (see below). Hence* $\boldsymbol{S}$ *is NOT a:*
  - *sphere with 1 or 2 punctures*
  - *1-gon (monogon) with 0 or 1 puncture*
  - *2-gon (digon) with 0 punctures*
  - *3-gon (triangle) with 0 punctures*

(n-gon means a disk with n marked points) We also exclude sphere with 3 punctures.

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Example 11.2. *An example:*

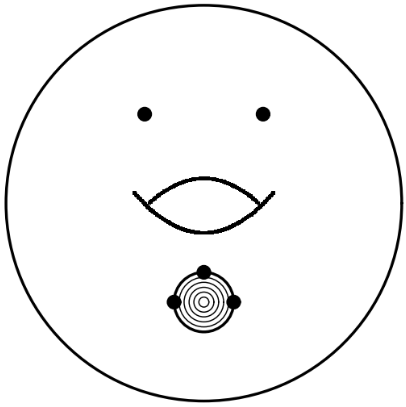


Figure 1: **S** is a torus with 2 punctures, 1 boundary components, and 5 marked points

We also have some restrictions on the arcs of triangulations:

Definition 11.3. *An arc γ in $(\boldsymbol{S}, \boldsymbol{M})$ is a curve up to isotopy, such that*

- *Endpoints of γ lie in $\boldsymbol{M}$*
- *γ does not self-intersect outside endpoints*
- *γ is disjoint from $\boldsymbol{M}$ and $\partial \boldsymbol{S}$ outside endpoints*
- *γ does not cut out an 1-gon or 2-gon with no punctures.*

Two arcs are compatible with they do not intersect in the interior of $\boldsymbol{S}$.
The set of all arcs is denoted by $\boldsymbol{A}^0(\boldsymbol{S}, \boldsymbol{M})$.

Definition 11.4. *An ideal triangulation is a maximal collection of distinct pairwise compatible arcs. The arcs cut $\boldsymbol{S}$ into ideal triangulations. (We allow self-folded triangles.)*

Each ideal triangulation consists of

$$n = 6g + 3b + 3p + c - 6$$

arcs, where

- *g=genus*
- *b=# boundary components*
- *p=# punctures*

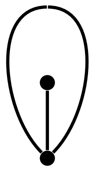


Figure 2: Self-folded triangle

- *c=# marked points on* $\partial \boldsymbol{S}$

Definition 11.5. *We can flip the triangles as usual. But edges inside a self-folded triangle cannot be flipped!*

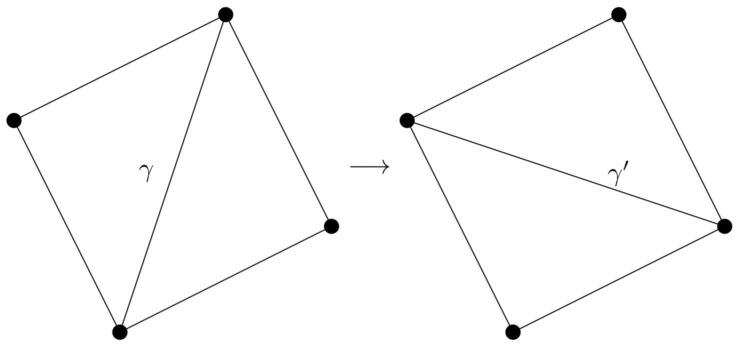


Figure 3: Usual flip

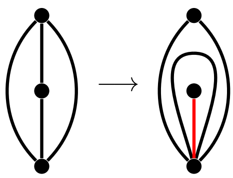


Figure 4: The red edge cannot be flipped

Definition 11.6. *The arc complex* $\Delta^0(\boldsymbol{S}, \boldsymbol{M})$ *is a simplicial complex with*

- *vertex = the arcs* $\in \boldsymbol{A}^0(\boldsymbol{S}, \boldsymbol{M})$,
- *simplex = compatible arcs*
- *maximal simpliecs = ideal triangulations*

The dual graph is $\boldsymbol{E}^0(\boldsymbol{S}, \boldsymbol{M})$: *vertices = triangulations, edges = flips*

Example 11.7. *Once-punctured triangle. We see that $\Delta^0(\boldsymbol{S}, \boldsymbol{M})$ has a boundary, and $\boldsymbol{E}^0(\boldsymbol{S}, \boldsymbol{M})$ is not 3-regular, since some edges cannot be flipped.*

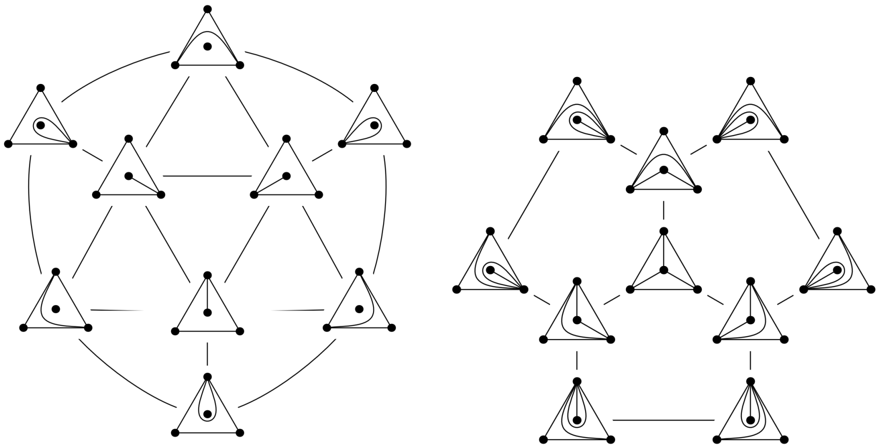


Figure 5: $\Delta^0(\mathbf{S}, \mathbf{M})$ and $\mathbf{E}^0(\mathbf{S}, \mathbf{M})$

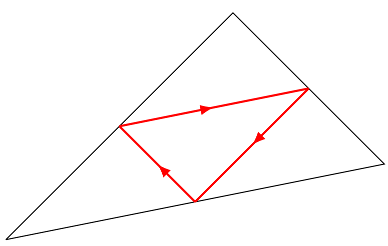
Proposition 11.8.
- *$\Delta^0(\boldsymbol{S}, \boldsymbol{M})$ is contratible unless $(\boldsymbol{S}, \boldsymbol{M})$ is a polygon without punctures, then it is $\simeq$ sphere S^n.*
- *$\boldsymbol{E}^0(\boldsymbol{S}, \boldsymbol{M})$ is connected. Hence we can obtain any triangulations by flipping.*
- *π_1 of $\boldsymbol{E}^0$ is generated by 4 and 5-cycles.*

Theorem 11.9. *There exists a triangluation for $(\boldsymbol{S}, \boldsymbol{M})$ with no self-folded triangles.*

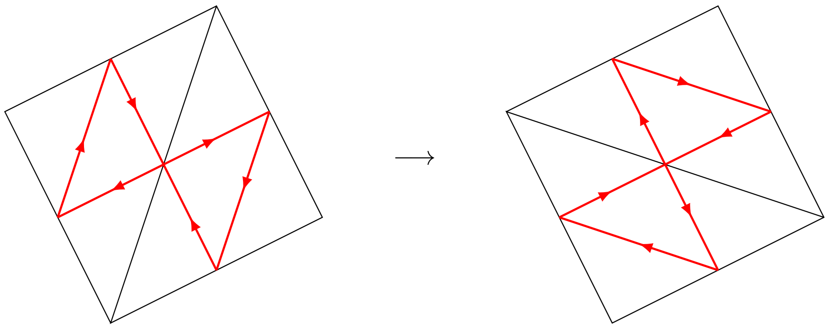
Hence we can always start with a nice triangulation, and obtain any other triangulations by flipping.

11.2 Cluster Algebra

Recall that given a triangulation (without self-folded triangles), we can associate to it a quiver Q, such that a flip correspond to quiver mutation:



Then for example, a change of triangulation will induce a change of quivers, called the "quiver mutaiton", that will be describe in more detail later.

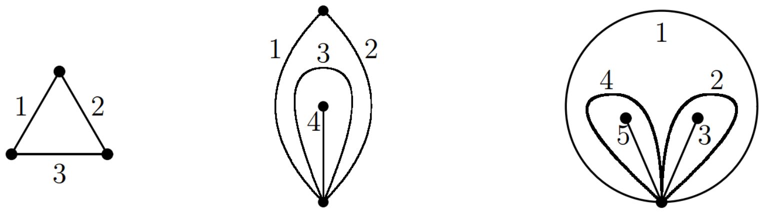


If $\widetilde{B}$ is the adjancency matrix of the quiver Q, then we can define a *cluster algebra of geometric type* $\mathcal{A}$ of rank n with initial seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ where $\mathbf{x} = (x_1, ..., x_n, x_{n+1}, ..., x_m)$

- n=number of arcs of the triangulation
- *cluster variables* $\mathbf{x} = (x_1, ..., x_n)$ labeled by the *arcs*,
- *frozen variables* $(x_{n+1}, ..., x_m)$ labeled by the *sides* of $\mathbf{S}$ (connected components of $\partial\mathbf{S} - \mathbf{M}$)

Remark 11.10. *In this quick overview, for simplicity, we will ignore the sides and only consider the principal part B of $\widetilde{B}$. The cluster algebra structures (cluster complex and exchange graph) are mostly independent of the coefficients (frozen variables). However, a major part of this theory deals with the properties and combinatorics of general coefficients (in $\mathbb{P}$) that is very interesting on its own.*

One can now define the adjancency matrix for a triangulation with self-folded triangles: they should be obtained from B by appropriate mutation. Any triangulation can be glued from 3 kind of "puzzle pieces", and B is obtained by summing up each matrix matching the row and column indices.



$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

Figure 6: The 3 puzzle pieces

Corollary 11.11. *By construction*

- $\Delta^0(\boldsymbol{S}, \boldsymbol{M})$ *is a subcomplex of the cluster complex of* $\mathcal{A}$.
- $\boldsymbol{E}^0(\boldsymbol{S}, \boldsymbol{M})$ *is a subgraph of the exchange graph of* $\mathcal{A}$.

In order to get the full cluster complex and exchange graph, we need to extend the definition of our triangulations to include tagged arcs.

11.3 Examples

Here we consider some examples:

Example 11.12. *Some examples in lower rank: (please work out the quivers...)*

- *4-gon, 0 punctures (type* A_1*)*
- *5-gon, 0 punctures (type* A_2*)*
- *2-gon, 1 puncture (type* $A_1 \times A_1$*)*
- *3-gon, 1 puncture (type* A_3*)*
- *torus, 1 puncture (Markov quiver)*

Example 11.13. *Polygons:*

- *Type* A_n*:* $(n+3)$*-gon with 0 punctures, from snake diagram*
- *Type* D_n*: n-gon with 1 puncture*

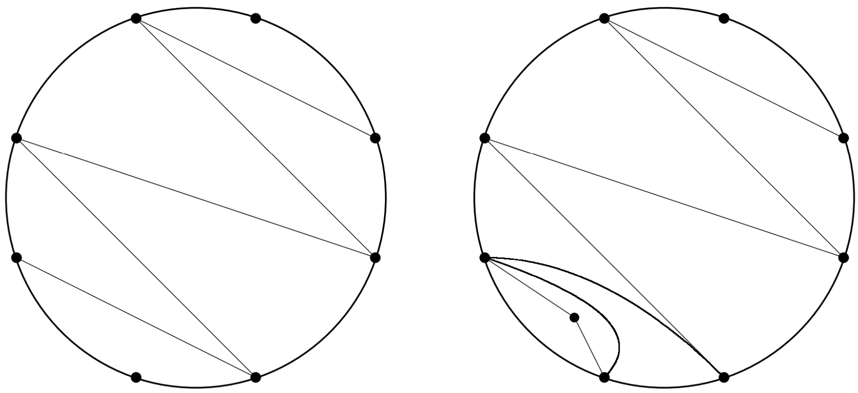


Figure 7: The snake diagram for type A_n and D_n

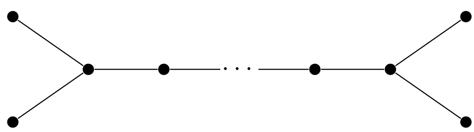
Also consider some special types:

Lemma 11.14. *Let Γ be n-cycle ($n \geq 3$) with n_1 edges in one direction and n_2 edges in another direction. Then the mutation equivalence class of Γ depends only on the unordered pair $\{n_1, n_2\}$.*

We call its type $\widetilde{A}(n_1, n_2)$. Note that $\widetilde{A}(n, 0) \simeq \widetilde{A}(0, n) \simeq D_n$ (See Lecture 6).

Also recall the affine diagram:

$\widetilde{D}_{n-1}$



Example 11.15.

- *Type $\widetilde{A}(n_1, n_2)$: Annulus with 0 puncture, n_1 marked points on one boundary, n_2 marked points on another.*
- *Type $\widetilde{A}(2, 2)$: 1-gon with 2 punctures*
- *Type $\widetilde{D}_{n-1}$: $(n-3)$-gon with 2 punctures.*

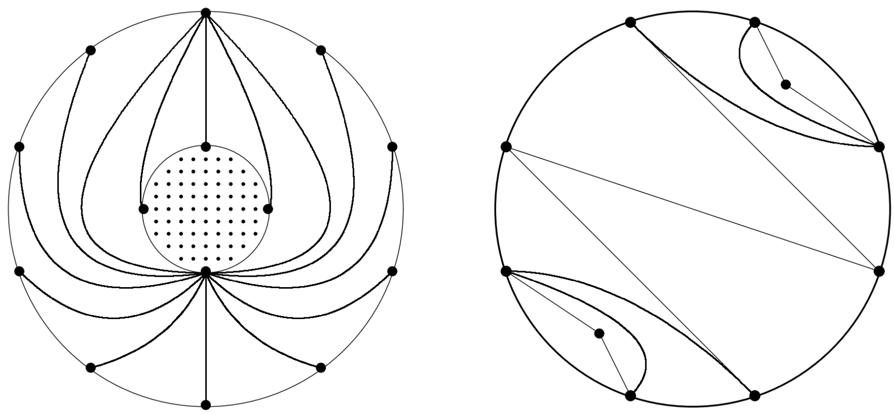


Figure 8: The quiver diagram for type $\widetilde{A}(10,4)$ and $\widetilde{D}_{10}$

11.4 Tagged arcs

We introduce tagging to resolve the self-folded triangles.

Definition 11.16. *A tagged arcs is an arc in $(\boldsymbol{S},\boldsymbol{M})$ with a tagging (plain or $\bowtie$) on each end*

- *Endpoint on $\partial\boldsymbol{S}$ is tagged plain*
- *Both ends of a loop are tagged the same way*
- *The arc does not cut out an 1-gon with 1 puncture*

The set of all tagged arcs is $\boldsymbol{A}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$. We let α_0 to be the untagged version of the tagged arc α.

Definition 11.17. *Replacement τ of an ordinary arc γ is a tagged arc $\tau(\gamma)$:*

- *if γ does not cut out an 1-gon with 1 puncture, then $\tau(\gamma)=\gamma$ with plain tags*
- *Otherwise if γ is a loop, it is given by the following replacement:*

Figure 9: γ and $\tau(\gamma)$

Hence there is a mpa from $\boldsymbol{A}^0(\boldsymbol{S},\boldsymbol{M}) \longrightarrow \boldsymbol{A}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$.

Definition 11.18. *Two tagged arcs* $\alpha, \beta \in \boldsymbol{A}^{\bowtie(\boldsymbol{S},\boldsymbol{M})}$ *are compatible if*

- *The untagged version* α_0, β_0 *are compatible*
- *If untagged version of* α_0, β_0 *are different, and they share an endpoint* a*, then they must be tagged the same way at* a.
- *If untagged version of* α_0, β_0 *are the same, they must be tagged the same way in at least one endpoint*

Definition 11.19. *Tagged arc complex* $\Delta^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *is the simplicial complex where*

- *vertex = the tagged arcs* $\in \boldsymbol{A}^0(\boldsymbol{S},\boldsymbol{M})$,
- *simplex = compatible tagged arcs*
- *maximal simpliecs = "tagged triangulations"*

Also let $\boldsymbol{E}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *be the dual graph. The edges of* $\boldsymbol{E}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *give us the "flipping of tagged triangulation".*

Remark 11.20. *If* $\boldsymbol{S}$ *has no punctures,* $\Delta^{\bowtie} = \Delta^0$ *and* $\boldsymbol{E}^{\bowtie} = \boldsymbol{E}^0$.

We can see now that we can extend our previous complexes:

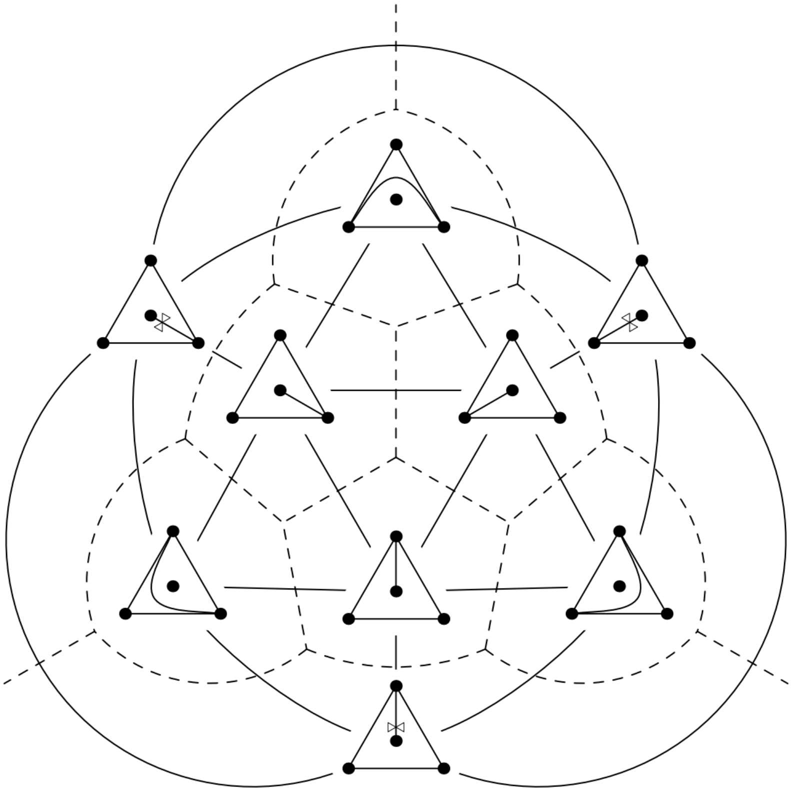


Figure 10: $\Delta^{\bowtie}(\mathbf{S},\mathbf{M})$ for 1 punctured triangle

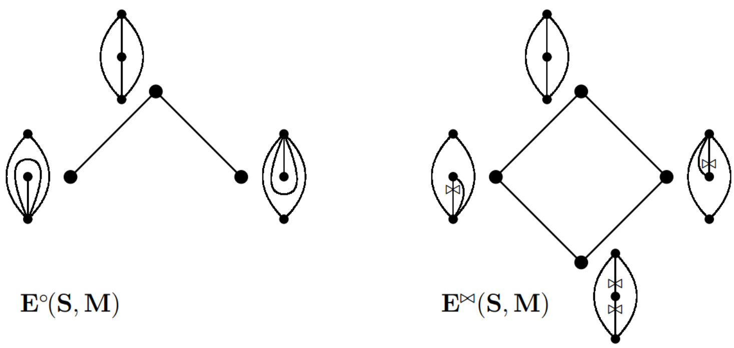


Figure 11: $\mathbf{E}^0(\mathbf{S},\mathbf{M})$ and $\mathbf{E}^{\bowtie}(\mathbf{S},\mathbf{M})$ for $\mathbf{S}$=2-gon with 1 puncture

Proposition 11.21.

- *If $(\boldsymbol{S},\boldsymbol{M})$ is not a closed surface with exactly one punc-*

ture, then $\boldsymbol{E}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *and* $\Delta^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *is connected.*

- *If* $(\boldsymbol{S},\boldsymbol{M})$ *is a closed surface with one puncture, then* $\boldsymbol{E}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *and* $\Delta^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *has two isomorphic components: one with all ends of arg tagged plain, and another with all ends of arg tagged* $\bowtie$*.*

We arrive at the main theorem:

Theorem 11.22. *Let* $\mathcal{A}$ *be the cluster algebra corresponding to* $(\boldsymbol{S},\boldsymbol{M})$*. Then*

- *If* $(\boldsymbol{S},\boldsymbol{M})$ *is not a closed surface with exactly one puncture, then* $\Delta(\mathcal{A}) \simeq \Delta^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$ *and exchange graph of* $\mathcal{A}$ *is* $\simeq \boldsymbol{E}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$*.*

- *If* $(\boldsymbol{S},\boldsymbol{M})$ *is a closed surface with exactly one puncture, then* $\Delta(\mathcal{A}) \simeq$ *a connected component of* $\Delta^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$*, and the exchange graph is* $\simeq$ *a connected component of* $\boldsymbol{E}^{\bowtie}(\boldsymbol{S},\boldsymbol{M})$*.*

To understand the idea behind the proof of the Theorem, we need to describe explicitly the tagged flipping as well as the adjacency matrix associated to a tagged triangulation.

Definition 11.23. *The undone version of a tagged triangulation* T *is an ordinary triangulation* T^0 *where*

- *if all arcs from a puncture is tagged* $\bowtie$*, remove the tag.*

- *for all other puncture, undo the map* τ *by replacing* γ *with a loop*

Proposition 11.24. *Tagged flipping has 2 types:*

(D) A "digon flip" as in Figure 11.

(Q) A "quadrilateral flip", which flips the corresponding undone version of the triangulation.

Example 11.25. *Illustrating (Q): sphere with 4 punctures.*

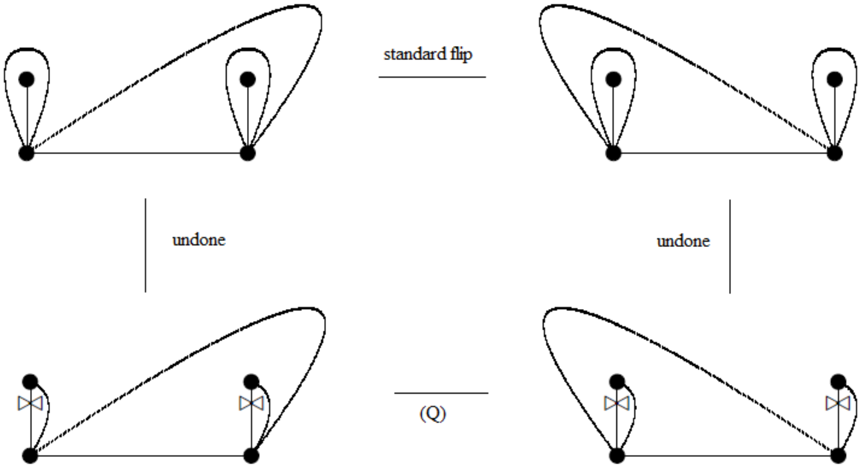


Figure 12: A Q flip

Proposition 11.26. *The adjacency matrix of a tagged triangulation T is defined as*

$$B(T) := B(T^0)$$

Then $B(T)$ satisfies the same matrix mutation rule with tagged flipping.

Corollary 11.27. *The cluster algebra from surface is of finite mutation type*

Proof. The adjacency matrix can only take value $0, \pm 1, \pm 2$. □

Example 11.28. *The adjancency matrix (quiver) associated with Figure 11:*

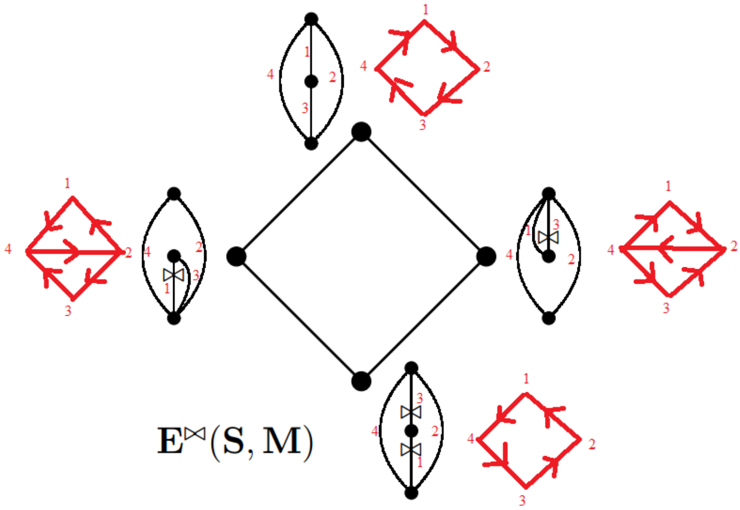


Figure 13: $\mathbf{E}^{\bowtie}(\mathbf{S}, \mathbf{M})$ for $\mathbf{S}$=2-gon with 1 puncture

11.5 Denominator Theorem

One can also describe the cluster variables explicitly, just like in the finite type. First we define the intersection number:

Definition 11.29. *The intersection number of 2 tagged arcs α, β is*

$$(\alpha|\beta) := A + B + C + D$$

where

- $A =$ *number of intersection of α_0 and β_0 outside the endpoints*
- $B = 0$ *unless α_0 is a loop based at a, with β_0 intersect α_0 at $\beta_1, ..., b_m$, then B is the $(-1)\times$ number of contractible triangle formed by b_i, b_{i+1}, a.*
- $C = \begin{cases} -1 & \alpha_0 = \beta_0 \\ 0 & \text{otherwise} \end{cases}$
- $D =$ *number of ends of β sharing an endpoint with α but tagged differently.*

From the Laurent phenomenon of cluster algebra, we also have an expression of the demonimator vectors:

Definition 11.30. *Fix an initial seed $(\mathbf{x}_0, B)$. Any $z \in \mathcal{A}$ can be written as a Laurent polynomial:*

$$z = \frac{P(\mathbf{x}_0)}{\prod_{x \in \mathbf{x}_0} x^{d(x|z)}}$$

where P is a polynomial in $\mathbf{x}_0$, and $d(x|z)$ is called the denominator vector.

Recall that each cluster variable correspond to a tagged arc $a \mapsto x[\alpha]$.

Theorem 11.31. *For any tagged arcs α, β, the denominator vector $d(x[\alpha]|x[\beta])$ equals the intersection number $(\alpha|\beta)$.*

References

[Surface-I] S. Fomin, M. Shapiro, and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, Acta Math. **201** (2008), 83146.

[Surface-II] S. Fomin, D. Thurston, *Cluster algebras and triangulated surfaces. Part II: Lambda lengths*, Memoirs of the American Mathematical Society (to appear), arXiv:1210.5569