Lecture Notes
Introduction to Cluster Algebra

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3 Definition and Examples of Cluster algebra

3.1 Quivers

We first revisit the notion of a quiver.

Definition 3.1. *A* quiver *is a finite oriented graph. We allow multiple arrows, but no 1-cycles and 2-cycles.*

1-cycle *2-cycle*

We will let some vertices be *frozen*, while others be *mutable*. We assume there are no arrows between frozen vertices.

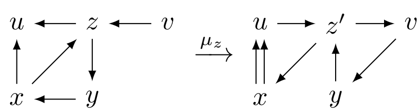
Definition 3.2. *Let k be mutable vertex in a quiver Q. A quiver mutation μ_k transforms Q into a new quiver $Q' := \mu_k(Q)$ by the three steps:*

(1) For each pair of directed edges $i \longrightarrow k \longrightarrow j$, introduce a new edge $i \longrightarrow j$ (unless both i, j are frozen)

(2) Reverse direction of all edges incident to k

(3) Remove all oriented 2-cycles.

Example 3.3. *Consider the example: (Let u, v be frozen)*



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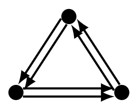
Proposition 3.4.
- *Mutation is an involution* $\mu_k(\mu_k(Q)) = Q$.
- *If* k *and* l *are two mutable vertices with no arrows between them, then the mutations at* k *and* l *commute* $\mu_l(\mu_k(Q)) = \mu_k(\mu_l(Q))$.

Definition 3.5. *Two quivers* Q *and* Q' *are called* mutation equivalent *if* Q *can be transformed into* Q' *by a sequence of mutations. The mutation equivalence class* $[Q]$ *is the set of all quivers which are mutation equivalent to* Q.

A quiver Q *is said to have* finite mutation type *if* $[Q]$ *is finite.*

Example 3.6.
- *All orientations of a tree are mutation equivalent to each other.*
- *The mutation equivalence class* $[Q]$ *of the* Markov quiver Q *consists of a single element.*



Definition 3.7. *Let* Q *be a quiver with* m *vertices, and* n *of them mutable. The* extended exchange matrix *of* Q *is the* $m \times n$ *matrix* $\widetilde{B}(Q) = (b_{ij})$ *defined by*

$$b_{ij} = \begin{cases} r & \text{if there are } r \text{ arrows from } i \text{ to } j \text{ in } Q \\ -r & \text{if there are } r \text{ arrows from } j \text{ to } i \text{ in } Q \\ -otherwise & \end{cases}$$

The exchange matrix $B(Q)$ *is the* $n \times n$ *skew-symmetric submatrix of* $\widetilde{B}(Q)$ *occupying the first* n *rows:*

$$B(Q) = (b_{ij})_{i,j \in [1,n]}$$

Lemma 3.8. *The extended exchange matrix* $\widetilde{B}' = (b'_{ij})$ *of the mutated quiver* $\mu_k(Q)$ *is given by*

$$b'_{ij} = \begin{cases} -b_{ij} & k \in \{i,j\} \\ b_{ij} + b_{ik}b_{kj} & b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{otherwise} \end{cases} \tag{3.1}$$

or more compactly:

$$b'_{ij} = \begin{cases} -b_{ij} & k \in \{i,j\} \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+ & \text{otherwise} \end{cases}.$$

or

$$b'_{ij} = \begin{cases} -b_{ij} & k \in \{i,j\} \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise} \end{cases}.$$

or

$$b'_{ij} = \begin{cases} -b_{ij} & k \in \{i,j\} \\ b_{ij} + sgn(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}.$$

Definition 3.9. *An $n \times n$ matrix B is* skew-symmetrizable*, if there exists integers $d_1, ..., d_n$ such that $d_i b_{ij} = -d_j b_{ji}$.*

An $m \times n$ integer matrix with top $n \times n$ submatrix skew-symmetrizable is called extended skew-symmetrizable *matrix.*

Definition 3.10. *The* diagram *of a skew-symmetrizable $n \times n$ matrix B is the weighted directed graph $\Gamma(B)$ such that there is a directed edge from i to j iff $b_{ij} > 0$, and this edge is assigned the weight $|b_{ij} b_{ji}|$.*

3.2 Cluster algebra of geometric type

Now we can define algebraically the notion of cluster algebra. We first define cluster algebra of geometric type (without coefficients). Let $m \geq n$ be two positive integers. Let the *ambient field* $\mathcal{F}$ be the field of rational functions over $\mathbb{C}$ in m independent variables.

Definition 3.11. *A labeled seed of geometric type in $\mathcal{F}$ is a pair $(\widetilde{\mathbf{x}}, \widetilde{B})$ where*

- *$\widetilde{\mathbf{x}} = (x_1, ..., x_m)$ is an m-tuple of elements of $\mathcal{F}$ forming a* free generating set*, i.e. $x_1, ..., x_m$ are algebraically independent, and $\mathcal{F} = \mathbb{C}(x_1, ..., x_m)$*
- *$\widetilde{B}$ is an $m \times n$ extended skew-symmetrizable integer matrix.*

We have the terminology:

- $\widetilde{\mathbf{x}}$ is the (labeled) *extended cluster* of the labeled seed $(\widetilde{\mathbf{x}}, \widetilde{B})$
- $\mathbf{x} = (x_1, ..., x_n)$ is the (labeled) *cluster* of this seed;
- $x_1, ..., x_n$ are its *cluster variables*;
- The remaining $x_{n+1}, ..., x_m$ of $\widetilde{\mathbf{x}}$ are the *frozen variables*;
- $\widetilde{B}$ is called the *extended exchange matrix* of the seed
- The top $n \times n$ submatrix B of $\widetilde{B}$ is the *exchange matrix*

"Labeled" means we also care about the order (index) of the seeds.

Definition 3.12. *A seed mutation μ_k in direction k transform the labeled seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ into a new labeled seed $(\widetilde{\mathbf{x}}', \widetilde{B}') := \mu_k(\widetilde{\mathbf{x}}, \widetilde{B})$ where $\widetilde{B}'$ is defined in (3.1), and $\widetilde{\mathbf{x}}' = (x_1', ..., x_m')$ is given by*

$$x_j' = x_j, \qquad j \neq k,$$

and $x_k' \in \mathcal{F}$ defined by the exchange relation

$$x_k x_k' = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{|b_{ik}|}$$

or equivalently

$$x_k x_k' = \prod_{i=1}^{m} x_i^{[b_{ik}]_+} + \prod_{i=1}^{m} x_i^{[-b_{ik}]_+}$$

We say two skew-symmetrizable matrices $\widetilde{B}$ and $\widetilde{B}'$ are mutation equivalent if one can get from $\widetilde{B}$ to $\widetilde{B}'$ by a sequence of mutations, possibly followed by simultaneous renumbering of rows and columns.

Definition 3.13. *Let $\mathbb{T}_n$ denote the n-regular tree. A seed pattern is defined by assigning a labeled seed $(\widetilde{\mathbf{x}}(t), \widetilde{B}(t))$ to every vertex $t \in \mathbb{T}_n$, so that the seeds assigned to the end points of any edge $t \xrightarrow{k} t'$ are obtained from each other by the seed mutation in direction k. To a seed pattern, we can associate an* exchange graph *which is n-regular, whose vertex are seeds and edges are mutations (the exchange graph is $\mathbb{T}_n$ only when no seeds repeat.)*

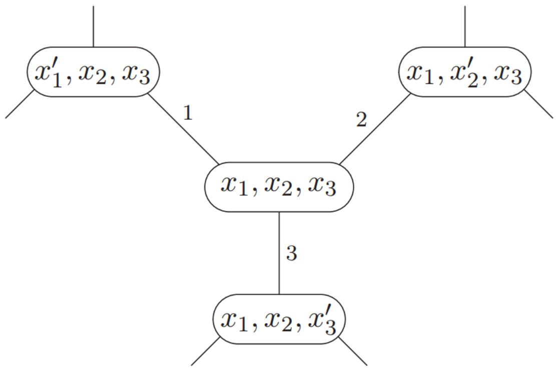


Figure 1: Exchange graph

Definition 3.14. *Let $(\widetilde{\mathbf{x}}(t), \widetilde{B}(t))$ be a seed pattern, and let*

$$\mathcal{X} := \bigcup_{t \in \mathbb{T}_n} \mathbf{x}(t)$$

be the set of all cluster variables appearing in its seeds. Let the ground ring $R = \mathbb{C}[x_{n+1}, ..., x_m]$ *be the polynomial ring generated by the frozen variables.*

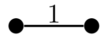
The cluster algebra of geometric type $\mathcal{A}$ *of* rank n *is the R-subalgebra of $\mathcal{F}$ generated by all cluster variables*

$$\mathcal{A} = R[\mathcal{X}]$$

Usually, we pick an initial seed $(\widetilde{\mathbf{x}}_0, \widetilde{B}_0)$, and build a seed pattern out of it. Then the corresponding cluster algebra $\mathcal{A}(\widetilde{\mathbf{x}}_0, \widetilde{B}_0)$ is generated over R by all cluster variables appearing in the seeds mutation equivalent to $(\widetilde{\mathbf{x}}_0, \widetilde{B}_0)$. Hence if we let $\mathcal{S}$ denote the set of all seeds, then we can write $\mathcal{A} = \mathcal{A}(\mathcal{S})$.

3.3 Examples

Rank 1 Case. $\mathbb{T}_1$ is very simple



We have two seeds and two clusters (x_1) and (x_1'). $\widetilde{B}_0$ can be any $m \times 1$ matrix with top entry 0. $\mathcal{A} \subset \mathcal{F} = \mathbb{C}(x_1, x_2, ..., x_m)$ is generated by $x_1, x_1', x_2, ..., x_m$ subject to relation of the form

$$x_1 x_1' = M_1 + M_2$$

where M_i are monomials in the frozen variables $x_2, ..., x_m$ which do not share a common factor.

Example 3.15. *$\mathbb{C}[SL_2] = \mathbb{C}[a, b, c, d]$ is a cluster algebra, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad = 1 + bc$. We have two extended clusters $\{a, b, c\}$ and $\{b, c, d\}$ and clusters $\{a\}$ and $\{d\}$.*

Example 3.16. *$\mathbb{C}[SL_3/N]$: Recall we have the Plücker relation*

$$\Delta_2 \Delta_{13} = \Delta_1 \Delta_{23} + \Delta_{12} \Delta_3$$

Then $\mathbb{C}[SL_3/N]$ has frozen variables $\{\Delta_1, \Delta_{12}, \Delta_{23}, \Delta_3\}$ and clusters $\{\Delta_2\}$, $\{\Delta_{13}\}$.

Rank 2 Case. Any 2×2 skew-symmetrizable matrix look like this:

$$\pm \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

for some positive integers b, c, or both zero. μ_1 or μ_2 simply changes its sign.

Example 3.17. *$b = c = 0$. This reduce to the rank 1 case.*

Example 3.18. *Let $\mathcal{A} = \mathcal{A}(b, c)$ denote cluster algebra of rank 2 with exchange matrix $\pm \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ and no frozen variables. Then we have*

$$x_{k+1} x_{k-1} = \begin{cases} x_k^c + 1 & k \text{ is even} \\ x_k^b + 1 & k \text{ is odd} \end{cases}$$

- *This is the same as the Conway-Coxeter frieze pattern for $(d_1, d_2) = (c, b)$.*
- *The exchange graph is finite only when $(d_1, d_2) = (1, 1), (1, 2), (1, 3)$, such that the graph is pentagon, hexagon and octagon respectively.*
- *In all other cases, the exchange graph is $\mathbb{T}_2$, which is an infinite line.*

Example 3.19. *Let us introduce frozen variable. Consider a seed pattern with initial seed $\{z_1, z_2, y\}$ and exchange matrix $\widetilde{B}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix}$. we get*

$$z_1, z_2, z_3 = \frac{z_2 + y^p}{z_1}, z_4 = \frac{y^{p+q} z_1 + z_2 + y^p}{z_1 z_2}, z_5 = \frac{y^q z_1 + 1}{z_2}, z_6 = z_1, z_7 = z_2$$

Again there are 5 distinct cluster variables. The cluster algebra is then defined to be

$$\mathcal{A} = R[\mathcal{X}] = \mathbb{C}[y^{\pm 1}]\left[z_1, z_2, \frac{z_2 + y^p}{z_1}, \frac{y^{p+q}z_1 + z_2 + y^p}{z_1 z_2}, \frac{y^q z_1 + 1}{z_2}\right]$$

Example 3.20. ***Grassmannian*** *Comparing the properties, we see that the coordinate ring of the Grassmannian $\mathbb{C}[Gr(2, n+3)]$ is a cluster algebra, where*

- *cluster* = $\{2\times\ 2$ *minors*$\}$ = *triangulation*
- *cluster variables* = $\Delta_{i,j}$ = *diagonals*
- *frozen variables* = $\Delta_{i,i+1}$ = *sides*
- *mutation* = *Plücker relation* = *flipping of diagonals*

Similarly, $\mathbb{C}[SL_n/N]$ is a cluster algebra.

Example 3.21. ***Markov triples.*** *Triples of integers satisfying*

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3.$$

Consider it as equation in x_1:

$$y^2 - (3x_2x_3)y + (x_2^2 + x_3^2) = 0.$$

Then it has two roots: $y = x_1$ and $x_1' = \frac{x_2^2+x_3^2}{x_1}$. Starting with $x_1 = x_2 = x_3 = 1$, replacing with another root: Vieta jumping.

$$\widetilde{B} = \pm \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

Conjecture 3.22 (Uniqueness)**.** *Maximal elements of Markov triples are all distinct.*

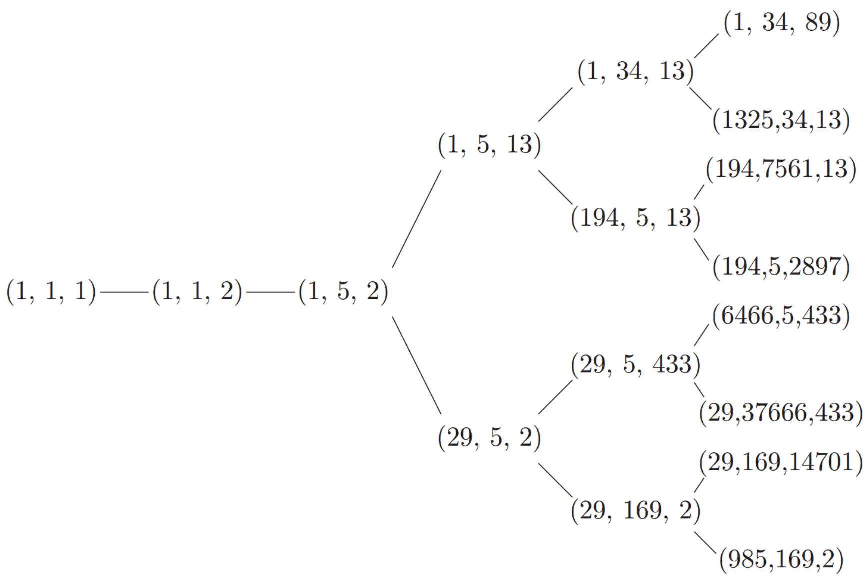
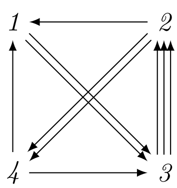


Figure 2: Markov triples

Example 3.23. ***Somos-4 sequence.***

$$x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}^2$$

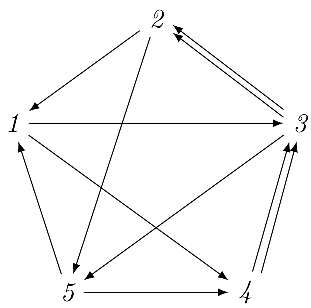
Mutate at 1 rotate the graph by 90 degree.



Somos-5 sequence.

$$x_n x_{n+5} = x_{n+1} x_{n+4} + x_{n+2} x_{n+3}$$

Mutate at 1 rotate the graph by 72 degree.



In both Somos-r sequence, x_n will be Laurent polynomials in the initial variables $x_1, ..., x_r$. In particular they will be integers if $x_1 = ... = x_r = 1$.
$\Longrightarrow$ *Laurent phenomenon!*

3.4 Semifields and coefficients

The mutation does not really use the frozen variables. So let us treat them as "coefficients", which leads to a more general notion of cluster algebra with semifields as coefficients.

Let us denote

$$y_j := \prod_{i=n+1}^{m} x_i^{b_{ij}}, \qquad j = 1, ..., n.$$

Then $y_1, ..., y_n$ encodes the same information as the lower $(n-m) \times n$ submatrix of $\widetilde{B}$. Hence a labeled seed can equivalently be presented as triples $(\mathbf{x}, \mathbf{y}, B)$ where $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n)$.

Now the mutation of x_k becomes:

$$\begin{aligned} x_k x_k' &= \prod_{i=n+1}^{m} x_i^{[b_{ik}]_+} \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=n+1}^{m} x_i^{[-b_{ik}]_+} \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \\ &= \frac{y_k}{y_k \oplus 1} \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \frac{1}{y_k \oplus 1} \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \end{aligned}$$

where the *semifield addition* is defined by

$$\prod_i x_i^{a_i} \oplus \prod_i x_i^{b_i} := \prod_i x_i^{\min(a_i, b_i)}$$

in particular,

$$1 \oplus \prod_i x_i^{b_i} := \prod_i x_i^{-[-b_i]_+}$$

The mutation of the frozen variables $x_{n+1}, ..., x_m$ also induces the mutation of the coefficient y-variables:

$$(y_1', ..., y_n') := \mu_k(y_1, ..., y_n)$$

$$y'_j := \begin{cases} y_k^{-1} & i = k \\ y_j(y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leq 0 \\ y_j(y_k^{-1} \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geq 0 \end{cases}$$

This is called *tropical Y-seed mutation rule* This is general, we can use any semifield!

Definition 3.24. *A semifield $(\mathbb{P}, \circ, \oplus)$ is an abelian group $(\mathbb{P}, \circ)$ (written multiplicatively) together with a binary operator $\oplus$ such that*

$$\oplus : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$
$$(p, q) \mapsto p \oplus q$$

is commutative, associative, and distributive:

$$p \circ (q \oplus r) = p \circ q \oplus p \circ r$$

Note: $\oplus$ may not be invertible!

Example 3.25. *Examples of semifield $(\mathbb{P}, \circ, \oplus)$:*

- $(\mathbb{R}_{>0}, \times, +)$
- $(\mathbb{R}, +, \min)$
- $(\mathbb{Q}_{sf}(u_1, ..., u_m), \cdot, +)$ *"subtraction-free" rational functions*
- $(Trop(y_1, ..., y_m), \cdot, \oplus)$ *Laurent monomials with usual multiplication and*

$$\prod_i x_i^{a_i} \oplus \prod_i x_i^{b_i} := \prod_i x_i^{\min(a_i, b_i)}$$

$\mathbb{P}$ is called the *coefficient group* of our cluster algbera.

Proposition 3.26. *If $(\mathbb{P}, \circ, \oplus)$ is a semifield, then*

- $(\mathbb{P}, \circ)$ *is torsion free (if there exists p, m such that $p^m = 1$, then $p = 1$).*
- *Let $\mathbb{Z}\mathbb{P}$ be the group ring of $(\mathbb{P}, \circ)$. Then it is a domain ($p \circ q = 0 \Longrightarrow p = 0$ or $q = 0$).*
- *Can define field of fractions $\mathbb{Q}\mathbb{P}$ of $\mathbb{Z}\mathbb{P}$*

Proof. (1) If $p^m = 1$, then note that $1 \oplus p \oplus ... \oplus p^{m-1} \in \mathbb{P}$ but $0 \notin \mathbb{P}$, we can write

$$p = p\frac{1 \oplus p \oplus ... \oplus p^{m-1}}{1 \oplus p \oplus ... \oplus p^{m-1}} = \frac{p \oplus p^2 \oplus ... \oplus p^m}{1 \oplus p \oplus ... \oplus p^{m-1}} = 1$$

(2) Let $p, q \in \mathbb{Z}\mathbb{P}$ with $p \circ q = 0$. Then p and q are contained in $\mathbb{Z}H$ for some finitely generated subgroup H of $\mathbb{P}$. Since $H \subset \mathbb{P}$ is abelian, $H \simeq \mathbb{Z}^n$, hence $\mathbb{Z}H \subset \mathbb{Z}(x_1, ..., x_n)$ consists of all Laurent polynomials in x_i. In particular $\mathbb{Z}H$ is an integral domain, and hence $p = 0$ or $q = 0$.

$\square$

Then we can set our ambient field to be $\mathcal{F} := \mathbb{QP}(u_1, ..., u_n)$. Now we can rewrite previous definitions and results:

Definition 3.27. *A labeled seed in $\mathcal{F}$ is $(\mathbf{x}, \boldsymbol{y}, B)$ with*

- $\mathbf{x} = \{x_1, ..., x_n\}$ *free generating set of $\mathcal{F}$*
- $\boldsymbol{y} = \{y_1, ..., y_n\} \subset \mathbb{P}$ *any elements*
- $B = n \times n$ *skew-symmetrizable $\mathbb{Z}$-matrix*

We have mutations for all $\mathbf{x}, \boldsymbol{y}$ and B, together with the exchange patterns.

A cluster algebra with coefficients $\mathbb{P}$ is then

$$\mathcal{A}(\mathbf{x}, \boldsymbol{y}, B) := \mathbb{ZP}\left[\bigcup_{t \in \mathbb{T}_n} \mathbf{x}(t)\right]$$

Example 3.28. *For rank $n = 2$ we have in the most general case:*

t	B_t	$\boldsymbol{y}_t$		$\mathbf{x}_t$	
0	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	y_1	y_2	x_1	x_2
1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$y_1(y_2 \oplus 1)$	$\frac{1}{y_2}$	x_1	$\frac{x_1y_2+1}{x_2(y_2\oplus 1)}$
2	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{1}{y_1(y_2\oplus 1)}$	$\frac{y_1y_2\oplus y_1\oplus 1}{y_2}$	$\frac{x_1y_1y_2+y_1+y_2}{(y_1y_2\oplus y_1\oplus 1)x_1x_2}$	$\frac{x_1y_2+1}{x_2(y_2\oplus 1)}$
3	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\frac{y_1\oplus 1}{y_1y_2}$	$\frac{y_2}{y_1y_2\oplus y_1\oplus 1}$	$\frac{x_1y_1y_2+y_1+x_2}{(y_1y_2\oplus y_1\oplus 1)x_1x_2}$	$\frac{y_1+x_2}{x_1(y_1\oplus 1)}$
4	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{y_1y_2}{y_1\oplus 1}$	$\frac{1}{y_1}$	x_2	$\frac{y_1+x_2}{x_1(y_1\oplus 1)}$
5	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	y_2	y_1	x_2	x_1

Remark 3.29. *The prevoius cluster algebra of geometric type = cluster algebra with coefficients $\mathbb{P} = Trop(x_{n+1}, ..., x_m)$.*

- We only need the $n \times n$ matrix B
- There are no frozen variables
- Mutation of $\mathbf{y}$ only involve two variables
- But we need to mutate all $\mathbf{y}$ variables, there are usually more $\mathbf{y}$ variables than cluster variables
- Y-pattern do not in general exhibit Laurent phenomenon

Definition 3.30. *Two cluster algebra* $\mathcal{A}(\mathcal{S})$ *and* $\mathcal{A}(\mathcal{S}')$ *are called* strongly isomorphic *if there exists a* $\mathbb{ZP}$*-algebra isomorphism* $\mathcal{F} \longrightarrow \mathcal{F}'$ *sending some seed in* $\mathcal{S}$ *into a seed in* $\mathcal{S}'$*, thus inducing a bijection* $\mathcal{S} \longrightarrow \mathcal{S}'$ *of seeds and an algebra isomorphism* $\mathcal{A}(\mathcal{S}) \longrightarrow \mathcal{A}(\mathcal{S}')$

Any cluster algebra $\mathcal{A}$ is uniquely determined by any single seed $(\mathbf{x}, \mathbf{y}, B)$. Hence $\mathcal{A}$ is determined by B and $\mathbf{y}$ up to strong isomoprhism, and we can write $\mathcal{A} = \mathcal{A}(B, \mathbf{y})$.