Lecture Notes
Introduction to Cluster Algebra

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5 Review of Root Systems

In this section, let us have a brief introduction to root system and finite Lie type classification using Dynkin diagrams. It follows [Fomin-Reading].

5.1 Reflection groups

Let V be Euclidean space.

Definition 5.1.
- *A reflection is a linear map $s : V \longrightarrow V$ that fixes a hyperplane, and reverse the direction of the normal vector of the hyperplane.*
- *A finite reflection group is a finite group generated by some reflections in V.*

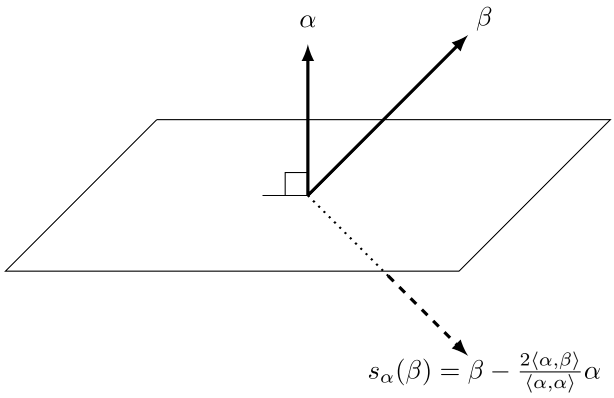

Figure 1: Orthogonal reflection

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Example 5.2. *Symmetry of the pentagon, given by the reflection group $I_2(5)$ (Dihedral group) generated by s and t. Products of odd number of generators are again reflection through some hyperplane (indicated in the picture). Products of even number of generators are rotations by certain angles.*

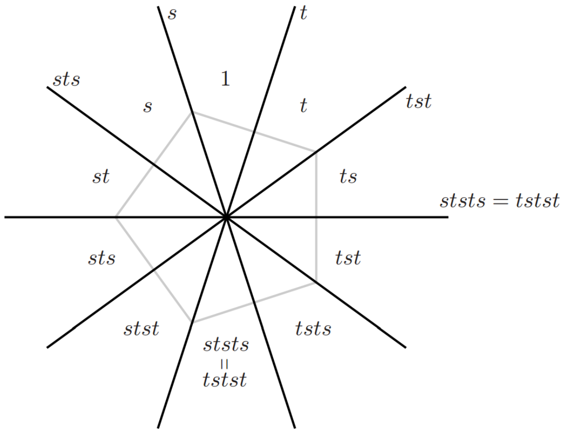


Figure 2: The reflection group $I_2(5)$

In general $I_2(m)$ is a group with $2m$ elements, generated by reflections s, t such that $(st)^m = 1$ and $s^2 = t^2 = 1$.

Definition 5.3. *The set $\mathcal{H}$ of all reflecting hyperplanes is called a* Coxeter arrangement*. It cuts V into connected components called* regions*.*

Lemma 5.4. *Fix an arbitrary region R_1. Then the map $w \mapsto R_w := w(R_1)$ is a bijection between reflection group W and the set of regions. Reflections by the facet hyperplanes of R_1 generates W.*

Let a hyperplane H_α given by $\{v : (v, \alpha) = 0\}$ for some vector α (normal vector). Let s_α be the reflection in the hyperplane H_α. Then

Lemma 5.5. *s_α is orthogonal linear transformation (i.e. preserving inner product), and*

$$s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

5.2 Root system

Definition 5.6. *A finite* root system *is a finite non-empty collection Φ of nonzero vectors in V called* roots *such that*

(1) *For all $\alpha \in \Phi$, $span(\alpha) \cap \Phi = \{\pm\alpha\}$.*

(2) If $\alpha \in \Phi$, then $s_\alpha(\Phi) = \Phi$. In particular if $\alpha \in \Phi$, then $-\alpha \in \Phi$.

Lemma 5.7. *Any reflection group W correpsond to a root system Φ_W: The roots correspond to the normal vector of the reflecting hyperplanes.*

Definition 5.8.
- *The* simple roots $\Pi \subset \Phi$ *are the roots normal to the facet hyperplanes of R_1 and pointing into the half-space containing R_1.*
- *The* rank *of Φ is n= dim(span(Φ)) $= \#\{$simple roots$\}$. $\Pi = \{\alpha_i : i \in I\}$ for some index set $I = [n] := \{1, 2, ..., n\}$.*
- *The set of positive roots Φ_+ (resp. negative roots Φ_-) are the roots $\alpha = \sum_{i\in I} c_i\alpha_i$ such that all $c_i \geq 0$. (resp. $c_i \leq 0$).*

Lemma 5.9. *Φ is disjoint union of Φ_+ and Φ_-.*

We also assume

(3) (Crystallographic condition) $s_\alpha(\beta) = \beta - c_{\alpha\beta}\alpha$ with $c_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

It means that any simple root coordinates of any root are integers. Crystallographic reflection group W is also called the *Weyl group* of Φ.

Lemma 5.10. $c_{\alpha\beta} \in \{0, \pm 1, \pm 2, \pm 3\}$.

Proof. Let the angle between α and β be θ. Then $(\alpha, \beta) = |\alpha||\beta| \cos\theta$, hence

$$c_{\alpha\beta} = 2\frac{|\beta|}{|\alpha|}\cos\theta \in \mathbb{Z}$$

and

$$c_{\alpha\beta}c_{\beta\alpha} = 4\cos\theta^2 \in \mathbb{Z}$$

which forces $|\cos\theta| = 0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$. If we assume $|\beta| \geq |\alpha|$, then $\frac{|\beta|}{|\alpha|} = 1, \sqrt{2}, \sqrt{3}$ respectively. (When $|\cos\theta| = 0$ there is no restriction. When $|\cos\theta| = 1, \beta = \pm\alpha$). □

Definition 5.11. *The* ambient space $Q_\mathbb{R}(\Phi) := \mathbb{R}$*-span*$(\Phi)$. *Root systems Φ and Φ' are* isomorphic *if there is an isometry of $Q_\mathbb{R}(\Phi) \longrightarrow Q_\mathbb{R}(\Phi')$ that sends Φ to some dilation $c\Phi'$ of Φ.*

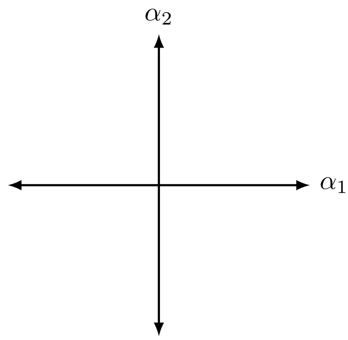
Definition 5.12. *The* Cartan matrix *of Φ is an integer matrix $C = (c_{ij})_{i,j\in I}$ where $c_{ij} := c_{\alpha_i\alpha_j} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ with $\alpha_i \in \Pi$.*

Lemma 5.13. *Root systems Φ and Φ' are isomorphic iff they have same Cartan matrix up to simultaneous rearrangement of rows and columns (i.e. reindexing).*

Example 5.14. *These are all 4 Cartan matrices of rank 2 and their corresponding root systems:*

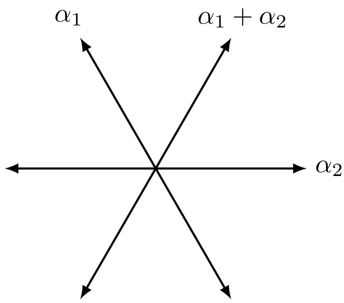
Type $A_1 \times A_1$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$



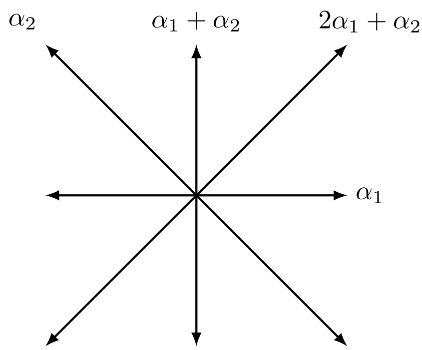
Type A_2

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

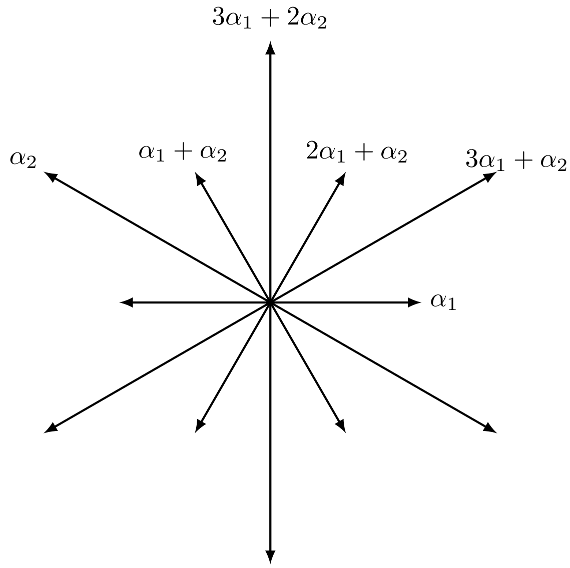


Type B_2

$$C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$



Type G_2
$C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$



They correspond to tiling of the plane.

Theorem 5.15. *An integer $n \times n$ matrix (c_{ij}) is a Cartan matrix of a root system iff*

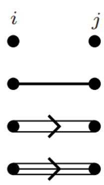
(1) $c_{ii} = 2$ *for every* i

(2) $c_{ij} \leq 0$ *if* $i \neq j$*, and* $c_{ij} = 0 \Longleftrightarrow c_{ji} = 0$

(3) There exists a diagonal matrix D such that DA is symmetric and positive definite (i.e. all eigenvalues > 0)

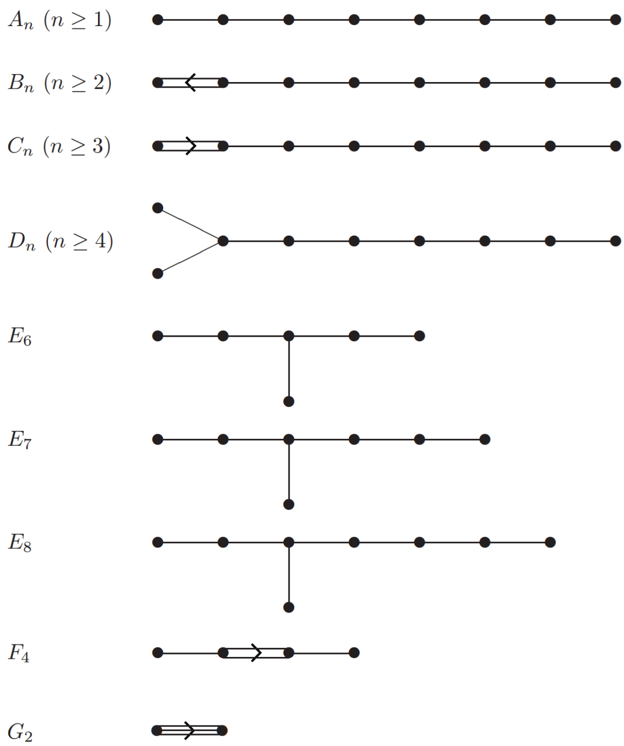
Definition 5.16. *A root system Φ is* reducible *if $\Phi = \Phi_1 \amalg \Phi_2$ such that $\alpha \in \Phi_1, \beta \in \Phi_2 \Longrightarrow (\alpha, \beta) = 0$, i.e. C is block diagonal with > 1 blocks. Otherwise it is* irreducible.

Definition 5.17. *Cartan matrix of finite type can be encoded by Dynkin diagrams:*



if $a_{ij} = a_{ji} = 0$
if $a_{ij} = a_{ji} = -1$
if $a_{ij} = -1$ and $a_{ji} = -2$
if $a_{ij} = -1$ and $a_{ji} = -3$

Theorem 5.18 (Cartan-Killing classification of irreducible root system)**.** *Any irreducible root system is isomorphic to the root system corresponding to the Dynkin diagram of $A_n (n \geq 1), B_n (n \geq 2), C_n (n \geq 3), D_n (n \geq 4), E_6, E_7, E_8, F_4$ or G_2.*



Example 5.19 (Root systems of type A_n)**.** *The root system is realized on the hyperplane* $P = \{x_1 + ... + x_n = 0\} \subset \mathbb{R}^{n+1}$*. Simple roots are realized as* $\alpha_i := e_{i+1} - e_i \in \mathbb{R}^{n+1}$ *and positive roots are* $e_i - e_j, 1 \leq j < i \leq n+1$*.*

Example 5.20 (Root systems of type B_n)**.** *Simple roots are realized as* $\alpha_1 = e_1, \alpha_i = e_i - e_{i-1}, i \geq 2$ *in* $\mathbb{R}^n$*. Positive roots are* e_i *and* $e_i \pm e_j$ *for* $1 \leq j < i \leq n$*.*

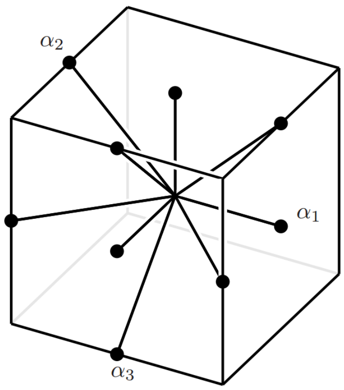


Figure 3: Roots system of type B_3

Example 5.21 (Root systems of type C_n). *Simple roots are realized as* $\alpha_1 = 2e_1, \alpha_i = e_i - e_{i-1}$ *in* $\mathbb{R}^n$. *Positive roots are* $2e_i$ *and* $e_i \pm e_j$ *for* $1 \leq j < i \leq n$. *The reflection group coincide with* B_n.

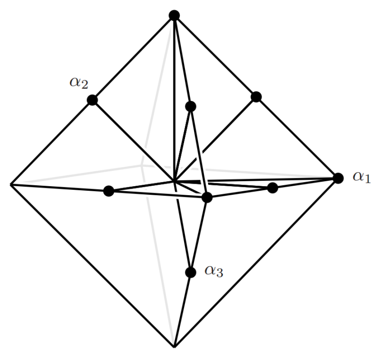


Figure 4: Roots system of type C_3

In type B_n and C_n, the action of W on the roots are not transitive, there are 2 orbits, corresponding to short and long roots.

Example 5.22 (Root system of type D_n). *Simple roots are realized as* $\alpha_0 = e_1 + e_2$ *and* $\alpha_i = e_{i+1} - e_i$. *Positive roots are* $e_i \pm e_j$ *for* $1 \leq j < i \leq n$.

5.3 Root Systems in Lie Theory

Root systems are related to representation theory of Lie algebra. Consider simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ (i.e. maximal commutative Lie subalgbera consisting of semisimple elements, i.e. diagonalizable in the adjoint representation).

Definition 5.23. *Let V be a representation of $\mathfrak{g}$. For any homomorphism $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$, i.e. $\lambda \in \mathfrak{h}^*$, we can define*

$$V_\lambda := \{v \in V : \forall \xi \in \mathfrak{h} : \xi \cdot v = \lambda(\xi)v\}$$

If V_λ is nonempty, then V_λ is called a weight space of V, and λ is called its weight. We have

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

Definition 5.24. *If V is the adjoint representation, i.e. $\mathfrak{g}$ acts on itself $V := \mathfrak{g}$ by $g \cdot v = [g, v]$, then the set of nonzero weights form a root system Φ. i.e. a root $\alpha \in \Phi$ is an element in $\mathfrak{h}^*$. In this case we have the root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Classification of root systems then give a classification of simple Lie algebra. In particular, $A_n = \mathfrak{sl}_{n+1}, B_n = \mathfrak{so}_{2n+1}, C_n = \mathfrak{sp}_{2n}$ and $D_n = \mathfrak{so}_{2n}$.

5.4 Some useful calculations

- $\Phi^+_{A_1} = \{\alpha\}$
- $\Phi^+_{A_2} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$
- $\Phi^+_{A_3} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$
- Let $s_i := s_{\alpha_i}$. Then $s_i(\alpha_i) = -\alpha_i$
- if $c_{ij} = -1$, then $s_i(\alpha_j) = \alpha_i + \alpha_j$
- if $c_{ij} = -2$, then $s_i(\alpha_j) = 2\alpha_i + \alpha_j$

Each element $w \in W$ can be written as product of simple reflections

$$w = s_{i_1} s_{i_2} ... s_{i_l}$$

Shortest factorization of this form is called a *reduced word* for w, and l is called the *length* of w.

Proposition 5.25. *Any Weyl group has a unique element w_0 of maximal length, called the* longest element

- $W_{A_1} = \{1, s_1\}$
- $W_{A_2} = \{1, s_1, s_2, s_1 s_2, s_2 s_1, w_0\}$, longest element is $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$
- $W_{A_3} = \mathcal{S}_4$ the permutation group. $w_0 = s_1 s_2 s_1 s_3 s_2 s_1$

- $W_{A_n} = \mathcal{S}_{n+1}$ is generated by transposition $s_i = (i, i+1)$. We have $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.
- $s_1 s_2 s_1 s_3 s_2 s_3$ is non-reduced: it equals $s_2 s_1 s_3 s_2$.
- In type A_2, $w_0(\alpha_1) = -\alpha_2, w_0(\alpha_2) = -\alpha_1$
- In type A_3, $w_0(\alpha_1) = -\alpha_3, w_0(\alpha_2) = -\alpha_2, w_0(\alpha_3) = -\alpha_1$
- In general, $w_0(\alpha_i) = -\alpha_{i^*}$. The map $i \mapsto i^*$ is called the Dynkin involution.

Definition 5.26. *Dynkin diagram has no cycles $\Longrightarrow$ bipartite. Let $I = I_+ \amalg I_-$ where the nodes are marked as $+$ and $-$ alternatively. A* Coxeter element *is defined as*

$$c = \left(\prod_{i \in I_+} s_i\right)\left(\prod_{i \in I_-} s_i\right) =: t_+ t_-$$

The order h of c (i.e. $c^h = 1$) is called Coxeter number*.*

Proposition 5.27. *The longest element w_0 can be written as*

$$w_0 = \underbrace{t_+ t_- t_+ t_- \ldots t_\pm}_{h}$$

In particular if $\mathfrak{g}$ is not of type A_{2n} then h is even and

$$w_0 = c^{\frac{h}{2}}.$$

Theorem 5.28. *We have the following table:*

Type	$\lvert\Phi_+\rvert$	h	$\lvert W\rvert$
A_n	$n(n+1)/2$	$n+1$	$(n+1)!$
B_n, C_n	n^2	$2n$	$2^n n!$
D_n	$n(n-1)$	$2(n-1)$	$2^{n-1} n!$
E_6	36	12	51840
E_7	63	18	2903040
E_8	120	30	696729600
F_4	24	12	1152
G_2	6	6	12