Lecture Notes
Introduction to Cluster Algebra

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9 Double Bruhat Cells

In this section, we describe the identification between double Bruhat cells and certain upper cluster algebra.

9.1 Notation and Definitions

Let $[1,r] := \{1,2,3,...,r\}$.

9.1.1 Lie Theory

- G a simply-connected, connected, semisimple complex algebraic group of rank r
- B, B_- the opposite Borel subgroups
- N, N_- the unipotent radicals
- $H = B \cap B_-$ a maximal torus
- $W = Norm_G(H)/H$ the Weyl group
- $\mathfrak{g} = Lie(G)$ the Lie algebra
- $\mathfrak{h} = Lie(H)$ the Cartan subalgebra
- A root system Φ and decomposition of root space: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$
- $\Pi = \{\alpha_1, ..., \alpha_r\} \subset \mathfrak{h}^*$ the simple roots, such that $\mathfrak{g}_{\alpha_i} \subset Lie(N)$.
- $\alpha_i^\vee \in \mathfrak{h}$ simple coroot, such that Cartan matrix is given by $c_{ij} = \alpha_j(\alpha_i^\vee)$

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- $\phi_i : SL_2 \longrightarrow G$ the embedding corresponding to $\mathfrak{sl}_2 \simeq \langle \mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i} \rangle \hookrightarrow \mathfrak{g}$
- The root subgroups are defined for $t \in \mathbb{C}$ by

  $x_i(t) := \phi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in N, \qquad h_i(t) := \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in H, \qquad x_{-i}(t) := \phi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in N_-$

9.1.2 Weyl group

Weyl group is generated by simple reflections s_i, and they can be represented by $s_i = \overline{s_i}H$ where

$$\overline{s_i} = \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Norm_G(H)$$

Can also be written as

$$\overline{s_i} = x_i(-1)x_{-i}(1)x_i(-1)$$

- In general for $w \in W$, we write

  $$\overline{w} \in Norm_G(H) \subset G$$

  for its representative in G given by products of $\overline{s_i}$.
- $\mathbf{i} = (i_1, ..., i_l)$ is called a reduced word for $w \in W$ if $w = s_{i_1}...s_{i_l}$ is a reduced expression.

In type A_r, $W \simeq \mathcal{S}_{r+1}$, and $\overline{s_i}$ is the permutation matrix (with appropriate signs).

9.1.3 $W \times W$

Now consider $W \times W$.

- We use $-1, ..., -r$ for the simple reflections on first copy of W, and $1, ..., r$ for second copy.
- A reduced word for $(u, v) \in W \times W$ is an arbitrary shuffle of reduced word for u written in $-[1, r]$ and reduced word for v written in $[1, r]$.
- Let $\epsilon(i)$ denote the sign of $i \in \pm[1, r]$.
- Let $Supp(u, v) = \{i : (u, v) \text{ contains either } i \text{ or } -i\} \subset [1, r]$ be the support.

9.1.4 Bruhat decompositions

Theorem 9.1. *The group G has Bruhat decompositions*

$$G = \coprod_{u \in W} BuB = \coprod_{v \in W} B_- v B_-$$

The double Bruhat cells are

$$G^{u,v} = BuB \cap B_- v B_-$$

Hence G is disjoint union of the double Bruhat cells.

Example 9.2. *When $G = SL_{r+1}(\mathbb{C})$, B (resp. B_-) can be chosen to be the upper (resp. lower) triangular matrix. Then Bruhat decompositions say that any element in G can be reduced to a permutation matrix after certain row and column operations.*

There is an explicit description of the Bruhat cells for type A_r:

Proposition 9.3. *For $G = SL_{r+1}(\mathbb{C})$, $x \in G$ belongs to BwB iff*

- $\Delta_{w([1,i]),[1,i])}(x) \neq 0$ *for* $i = 1, ..., r$
- $\Delta_{w([1,i-1]\cup\{j\}),[1,i]}(x) = 0$ *for* $i < j$ *and* $w(i) < w(j)$

This follows from the explicit description below in Prop 9.19.

Example 9.4. *In particular for $G = SL_3(\mathbb{C})$, let $u = v = w_0 = s_1s_2s_1 = s_2s_1s_2$. Then $G^{w_0,w_0} \subset SL_3(\mathbb{C})$ consists of 3×3 matrices x with $\det(x) = 1$ such that*

$$x_{13} \neq 0, \qquad x_{31} \neq 0, \qquad \det\begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix} \neq 0, \qquad \det\begin{pmatrix} x_{12} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \neq 0$$

The following Theorem gives important structural properties of the double Bruhat cells:

Theorem 9.5.

- *$G^{u,v}$ is biregularly isomoprhic to a Zariski open subset of an affine space of dimension $r + l(u) + l(v)$.*
- *Let $\mathbf{i} = (i_1, ..., i_N)$ be a reduced word for $(u, v) \in W \times W$. Define the map $x_{\mathbf{i}} : H \times \mathbb{C}^N \longrightarrow G$ by*

$$x_{\mathbf{i}}(a; t_1, ..., t_N) = a x_{i_1}(t_1) ... x_{i_N}(t_N)$$

Then the map $x_{\mathbf{i}}$ restricts to a biregular isomorphism between $H \times \mathbb{C}^N_{\neq 0}$ and a Zariski open subset of the double Bruhat cell $G^{u,v}$.

Proof. Let us show that $x_{\mathbf{i}}(H \times \mathbb{C}^N_{\neq 0}) \subset B_- v B_-$. Consider the part of the words $(i_{k_1}, ..., i_{k_l})$ that form a reduced word for v (i.e. all those with $\epsilon = +$). Note that we have

$$x_i(t) \in B_- s_i B_-, \qquad x_{-i}(t) \in B_-$$

Hence

$$\begin{aligned} x_{\mathbf{i}}(a; t_1, ..., t_N) &\in B_- \cdot B_- s_{i_{k_1}} B_- \cdot B_- \cdots B_- \cdot B_- s_{i_{k_l}} B_- \cdot B_- \\ &= B_- v B_- \end{aligned}$$

since it is well-known that

$$B_- w' B_- \cdot B_- w'' B_- = B_- w' w'' B_-$$

whenever $l(w'w'') = l(w') + l(w'')$.

It is also easy to see that this map is injective. □

Example 9.6. *Consider $G = SL_3(\mathbb{C})$. Consider $u = s_1s_2, v = e$. The image in $G^{s_1s_2,e} = Bs_1s_2B \cap B_-$ is*

$$x_{(1,2)}(a;t_1,t_2) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2t_1 & a_2 & 0 \\ 0 & a_3t_2 & a_3 \end{pmatrix}$$

where $a_1a_2a_3 = 1$. We see that $Bs_1s_2B \cap B_-$ can be described by

$$\Delta_{3,1} = 0, \qquad \Delta_{2,1} \neq 0, \qquad \Delta_{3,2} \neq 0$$

Furthermore, we can recover our parameters from some minors (say, with consecutive columns) by monomial transforms, e.g.

$$a_1 = \Delta_{1,1}, a_2 = \frac{\Delta_{12,12}}{\Delta_{1,1}}, t_1 = \frac{\Delta_{1,1}\Delta_{2,1}}{\Delta_{12,12}}, t_2 = \frac{\Delta_{12,12}\Delta_{23,12}}{\Delta_{2,1}}$$

Example 9.7. *On the other hand, the image in $G^{s_2s_1,e} = Bs_2s_1B \cap B_-$ is*

$$x_{(1,2)}(a;t_1,t_2) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2t_2 & a_2 & 0 \\ a_3t_1t_2 & a_3t_1 & a_3 \end{pmatrix}$$

We see that in this case $Bs_2s_1B \cap B_-$ can be described by

$$\Delta_{23,12} = 0, \qquad \Delta_{3,1} \neq 0, \qquad \Delta_{3,2} \neq 0.$$

and similarly we can recover our parameters by monomial transforms:

$$a_1 = \Delta_{1,1}, a_2 = \frac{\Delta_{12,12}}{\Delta_{1,1}}, t_1 = \frac{\Delta_{12,12}\Delta_{13,12}}{\Delta_{1,1}}, t_2 = \frac{\Delta_{1,1}\Delta_{2,1}}{\Delta_{12,12}}$$

In general, the Bruhat cells can be described by conditions of the form $\Delta(x) = 0, \Delta(x) \neq 0$. (see Proposition 9.19) and the parameters can be recovered by certain collection $F(\mathbf{i})$ of minors (see Definition 9.21). More precisely,

$$H \times \mathbb{C}^m_{\neq 0} \simeq_{monomial} G^{u^{-1},v^{-1}} \simeq_{twisting} G^{u,v}$$

for some twisting $G^{u,v} \longrightarrow G^{u^{-1},v^{-1}}$ which is a biregular isomorphism, and the parameters of x' can be expressed as Laurent monomials from the minors in $F(\mathbf{i})$.

9.2 Combinatorial data

From a reduced word $\mathbf{i} = (i_1, ..., i_{l(u)+l(v)})$, we will construct a rectangular matrix $\widetilde{B} = \widetilde{B}(\mathbf{i})$ which defines our cluster algebra of geometric type.

Let

$$M := -[1, r] \cup [1, l(u) + l(v)]$$

and let $m := |M| = r + l(u) + l(v)$.

- Let us add $i_{-r}, ..., i_{-1}$ at the beginning of $\mathbf{i}$ by setting $i_{-j} = -j$ for $j \in [1, r]$
- For $k \in M$, let k^+ be smallest index l such that $k < l$ and $|i_l| = |i_k|$. If it does not exist we set $k^+ = l(u) + l(v) + 1$.
- k is called $\mathbf{i}$-exchangeable if both $k, k^+ \in [1, l(u), l(v)]$.
- Let $\mathbf{e}(\mathbf{i})$ be the set of $\mathbf{i}$-exchangeable indices. Let

$$n := |\mathbf{e}(\mathbf{i})| = l(u) + l(v) - |Supp(u, v)|$$

- Let $\widetilde{B}(\mathbf{i})$ be a $m \times n$ matrix. The rows are labeled by the set M and columns are labeled by the set $\mathbf{e}(\mathbf{i})$

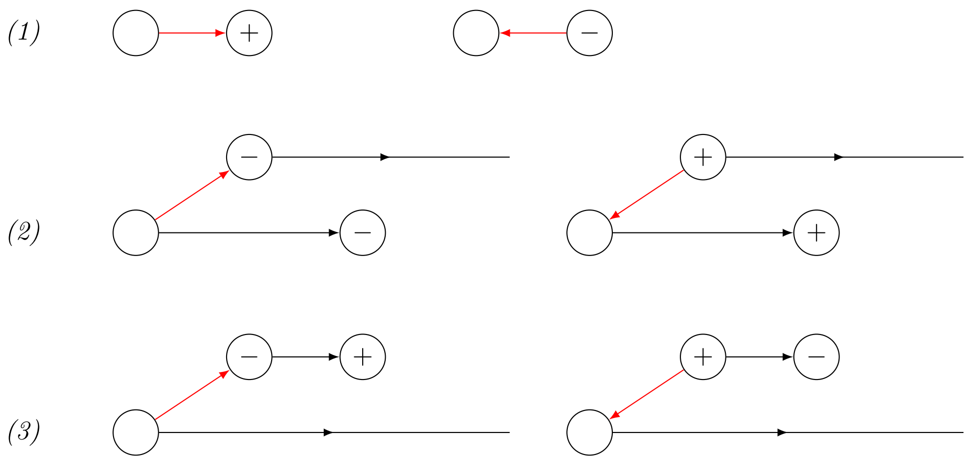
Definition 9.8. *The quiver $\Gamma(\mathbf{i})$ has vertices set M. For vertices k, l with $k < l$, it is connected by an edge iff either k or l (or both) are $\mathbf{i}$-exchangeable, and*

(1) $l = k^+$

(2) $l < k^+ < l^+, c_{|i_k|,|i_l|} \neq 0$ *and* $\epsilon(i_l) = \epsilon(i_{k^+})$

(3) $l < l^+ < k^+, c_{|i_k|,|i_l|} \neq 0$ *and* $\epsilon(i_l) = -\epsilon(i_{l^+})$

- *In case (1), the (horizontal) edge is $k \longrightarrow l$ if $\epsilon(i_l) = +$, and vice versa.*
- *In case (2) and (3), the (inclined) edge is $k \longrightarrow l$ if $\epsilon(i_l) = -$, and vice versa.*



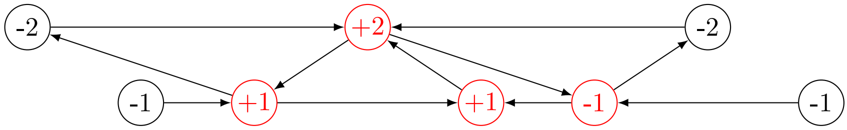


Figure 1: $\Gamma(\mathbf{i})$ for $SL_3^{w_0,w_0}$

Definition 9.9. *The matrix $\widetilde{B}$ is defined by*

(1) $b_{kl} = 0$ iff there are no edges connecting k and l.

(2) $b_{kl} > 0$ if $k \longrightarrow l$, $b_{kl} < 0$ if $k \longleftarrow l$

(3) $|b_{kl}| = \begin{cases} 1 & |i_k| = |i_l| \quad \textit{(horizontal edge)} \\ -c_{|i_k|,|i_l|} & |i_k| \neq |i_l| \quad \textit{(inclined edge)} \end{cases}$

Example 9.10. *For $G = SL_3$, $r = 2$. Take $u = v = w_0$. We have $l(u) = l(v) = 3$, $m = 8, n = 4$. Take $\mathbf{i} = (1, 2, 1, -1, -2, -1)$. Then $\mathbf{e}(\mathbf{i}) = \{1, 2, 3, 4\}$. The graph has vertices $\{-2, -1, 1, 2, 3, 4, 5, 6\}$ but we label them by $\mathbf{i}$. The vertices $\in \mathbf{e}(\mathbf{i})$ is highlighted in red. Then $\widetilde{B}$ is given by*

	1	2	3	4
-2	-1	1	0	0
-1	1	0	0	0
1	0	-1	1	0
2	1	0	-1	1
3	-1	1	0	-1
4	0	-1	1	0
5	0	1	0	-1
6	0	0	0	1

with the (skew-symmetric) principal part B highlighted in red.

Proposition 9.11. *The matrix $\widetilde{B}(\mathbf{i})$ has full rank n. Its principal part $B(\mathbf{i})$ is skew-symmetrizable.*

Proof. Enough to show that the determinant of the $n \times n$ submatrix Δ of $\widetilde{B}$ labeled by the row set

$$\mathbf{e}(\mathbf{i})^- := \{k \in M : k^+ \in \mathbf{e}(\mathbf{i})\}$$

is nonzero. Note that if $k \in \mathbf{e}(\mathbf{i})^-$ and $l \in \mathbf{e}(\mathbf{i})$, then

$$|b_{kl}| = \begin{cases} 1 & k^+ = l \\ 0 & k^+ < l \end{cases},$$

hence Δ is triangular (after reindexing) with diagonal entries $\neq 0$.

The matrix Δ (-1 and -2 interchanged):

	1	2	3	4
-1	1	0	0	0
-2	-1	1	0	0
1	0	-1	1	0
3	-1	1	0	-1

Cartan matrix is symmetrizable $\Longrightarrow B(\mathbf{i})$ is skew-symmetrizable by our definition.

□

Hence by previous results, the matrix $\widetilde{B}(\mathbf{i})$ give rise to a well-defined upper cluster algebra $\overline{\mathcal{A}(\mathbf{i})}$ of geometric type, which coincides with the upper bound $\mathcal{U}(\mathcal{S})$ for the seed $\mathcal{S}(\mathbf{i}) = (\mathbf{x}, \widetilde{B}(\mathbf{i}))$. The ambient field $\mathcal{F}$ of $\overline{\mathcal{A}}(\mathbf{i})$ is the field of rational functions over $\mathbb{Q}$ in m independent variables $\widetilde{\mathbf{x}} = \{x_k : k \in M\}$. The cluster variables in $\mathbf{x}$ are labeled by the set $\mathbf{e}(\mathbf{i})$, and the coefficient group $\mathbb{P}$ is generated by the remaining indices.

Example 9.12. *In our previous example for* $\mathbf{i} = (-2, -1, 1, 2, 1, -1, -2, -1)$*, we have* $\mathbf{x} = \{x_1, x_2, x_3, x_4\}$*,* $\mathbb{P} = \langle x_{-2}^{\pm}, x_{-1}^{\pm}, x_5^{\pm}, x_6^{\pm}\rangle$*, and the exchange relation*

$$\begin{aligned} x_1 x_1' &= x_{-1}x_2 + x_{-2}x_3 \\ x_2 x_2' &= x_{-2}x_3x_5 + x_1x_4 \\ x_3 x_3' &= x_1x_4 + x_2 \\ x_4 x_4' &= x_2x_6 + x_3x_5 \end{aligned}$$

The algebra $\overline{\mathcal{A}}(\mathbf{i})$ *consists of all rational functions in* $\mathcal{F} = \mathbb{Q}(x_{-2}, x_{-1}, x_1, ..., x_6)$ *that can be written as Laurent polynomials in each of the 5 clusters:*

$$\mathbf{x} = (x_1, x_2, x_3, x_4), \mathbf{x}_1 = (x_1', x_2, x_3, x_4), \cdots, \mathbf{x}_4(x_1, x_2, x_3, x_4')$$

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