Lecture Notes
Introduction to Cluster Algebra

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9.3 Generalized minors

Consider the weight lattice $P \subset \mathfrak{h}^*$ of G given by weights $\gamma \in \mathfrak{h}^*$ such that $\gamma(\alpha_i^\vee) \in \mathbb{Z}$ for all $i = 1, ..., r$. P has a basis given by the fundamental weights $\{\omega_1, ..., \omega_r\}$ such that

$$\omega_j(\alpha_i^\vee) = \delta_{ij}.$$

$\gamma \in \mathfrak{h}^*$ can be treated as multiplicative characters $H \longrightarrow \mathbb{C}$ written as

$$a \mapsto \gamma(a) =: a^\gamma \in \mathbb{C}, \qquad a \in H$$

Definition 9.13. *Let $G_0 = N_- HN$ be the open subset of elements $x \in G$ that have Gaussian decomposition. We write*

$$x = [x]_-[x]_0[x]_+, \qquad [x]_- \in N_-, [x]_0 \in H, [x]_+ \in N$$

Definition 9.14. *Let Δ^{ω_i} be a regular function on G whose restriction to G_0 is given by*

$$\Delta^{\omega_i}(x) := [x]_0^{\omega_i}.$$

The generalized minor $\Delta_{u\omega_i, v\omega_i}$ is the regular function on G whose restriction to the open set $\overline{u}G_0\overline{v}^{-1}$ is given by

$$\Delta_{u\omega_i, v\omega_i}(x) = \Delta^{\omega_i}(\overline{u}^{-1}x\overline{v})$$

By definition we have for any $x \in G, n^- \in N_-, n^+ \in N, a \in H$:

$$\begin{aligned} \Delta^{\omega_i}(n^-x) = \Delta^{\omega_i}(xn^+) = \Delta^{\omega_i}(x) \\ \Delta^{\omega_i}(ax) = \Delta^{\omega_i}(xa) = a^{\omega_i}\Delta^{\omega_i}(x) \end{aligned} \tag{9.1}$$

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Remark 9.15. *There is a representation theoretic meaning to the generalized minors. The above formula says that Δ^{ω_i} is invariant under the action of N^+ on the right, and is an eigenvector for the action of H. Hence this says that $\Delta^{\omega_i} \in \mathbb{C}[G]$ is a highest weight vector of weight ω_i, where G acts on $\mathbb{C}[G]$ by right translations. Some results can be proved using this language:*

Lemma 9.16. *$\Delta_{u\omega_i, v\omega_i}$ depends only on the weights $u\omega_i, v\omega_i$ and not on the choice of u or v.*

Proof. We consider the case for v. The case for u is similar. Recall that $s_j(w_i) = w_i$ for $i \neq j$. Hence we only need to show for $i \neq j$,

$$\Delta^{\omega_i}(x\overline{s_j}) = \Delta^{\omega_i}(x).$$

Recall $\overline{s_i} = x_i(-1)x_{-i}(1)x_i(-1)$. Since Δ^{ω_i} is highest weight vector of weight ω_i, it is trivial with respect to $\phi_j(SL_2)$ since $\omega_i(h_j) = 0$. Therefore we have

$$\Delta^{\omega_i}(xx_{-j}(t)) = \Delta^{\omega_i}(x)$$

and also by (9.1) gives the claim. □

Proposition 9.17. *Δ^{ω_i} can be extended from G_0 to the whole G by*

$$\Delta^{\omega_i}(x) = \begin{cases} a^{\omega_i} & w\omega_i = \omega_i \\ 0 & \textit{otherwise} \end{cases}$$

where $x = x^- a\overline{w}x^+$ for some $x_- \in N_-, a \in H, w \in W, x^+ \in N$ by the Bruhat decomposition.

In particular, $x \in G_0$ (i.e. $w = e$) iff $\Delta^{\omega_i}(x) \neq 0$ for any $i \in [1, r]$.

Proof. Since $\Delta^{\omega_i}(x^- a\overline{w}x^+) = a^{\omega_i}\Delta^{\omega_i}(\overline{w})$, only need to show

$$\Delta^{\omega_i}(\overline{w}) = \begin{cases} 1 & w\omega_i = \omega_i \\ 0 & \textit{otherwise} \end{cases}$$

This is done by induction on $l(w)$ and direct calculation. □

Example 9.18. *In type A_r, $\Delta_{u\omega_i, v\omega_i}(x)$ is the determinant of the submatrix of x whose row (resp. columns) are labeled by elements of the set $u([1, i])$ (resp. $v([1, i])$) where $u, v \in W \simeq \mathcal{S}_{r+1}$.*

Each (double) Bruhat cell can be defined inside G by a collection of conditions of the form $\Delta(x) = 0$ and $\Delta(x) \neq 0$ where Δ is a generalized minor. It can be described explicitly as

Theorem 9.19. *The Bruhat cell $BuB \subset G$ is given by conditions*

- $\Delta_{u'\omega_i,\omega_i} = 0$ *whenever* $u'\omega_i \not\leq u\omega_i$ *in the Bruhat order*
- $\Delta_{u\omega_i,\omega_i} \neq 0$

Similarly the Bruhat cell $B_-vB_- \subset G$ *is given by conditions*

- $\Delta_{\omega_i,v'\omega_i} = 0$ *whenever* $v'\omega_i \not\leq v^{-1}\omega_i$ *in the Bruhat order*
- $\Delta_{\omega_i,v^{-1}\omega_i} \neq 0$

where the Bruhat order on weights $w\omega_i$ *are induced from the Bruhat order on* W. *(Bruhat order on* W*: for* $u, v \in W$, $u < v \iff l(v) = l(u) + l(u^{-1}v)$*)*

Example 9.20. *In type* A_r, $u'\omega_i \leq u\omega_i \iff u'([1,i]) \leq u([1,i])$ *where the partial order on* i*-element set is defined by*

$$\{j_1 < \cdots < j_i\} \leq \{k_1 < \cdots < k_i\} \iff (j_1 \leq k_1, \cdots, j_i \leq k_i)$$

Next we introduce the family of minors that should be grouped as cluster.

Definition 9.21. *We define elements* $u_{\leq k} \in W, v_{>k} \in W$ *by*

$$u_{\leq k} := u_{\leq k}(\mathbf{i}) := \prod_{\substack{l=1,\dots,k \\ \epsilon(i_l)=-}} s_{|i_l|}$$

$$v_{>k} := v_{>k}(\mathbf{i}) := \prod_{\substack{l=l(u)+l(v),\dots,k+1 \\ \epsilon(i_l)=+}} s_{|i_l|}$$

(The product is increasing (resp. decreasing) in l *for* $u_{\leq j}$ *(resp.* $v_{>k}$*). (i.e. they are truncated words for* u *and* v*)*

For $k \in -[1,r]$, *we define* $u_{\leq k} = e, v_{>k} = v^{-1}$. *For* $k \in M$, *we set*

$$\Delta(k;\mathbf{i}) := \Delta_{u_{\leq k}\omega_{i_k}, v_{>k}\omega_{|i_k|}}$$

and define

$$F(\mathbf{i}) := \{\Delta(k;\mathbf{i}) : k \in M\}$$

the set of $m = r + l(u) + l(v)$ *minors associated with the fixed reduced word* $\mathbf{i}$.

Example 9.22. *Continuing our example for* $G = SL_3$, $u = v = w_0$ *and* $\mathbf{i} = (-2,-1,1,2,1,-1,-2,-1)$. *Recall* $u\omega_i := u([1,i])$.

k	-2	-1	1	2	3	4	5	6
i_k	-2	-1	1	2	1	-1	-2	-1
$u_{\leq k}$	e	e	e	e	e	s_1	s_1s_2	$s_1s_2s_1$
$v_{>k}$	$s_1s_2s_1$	$s_1s_2s_1$	s_1s_2	s_1	e	e	e	e
$u_{\leq k}[1,\lvert i_k\rvert]$	12	1	1	12	1	2	23	3
$v_{>k}[1,\lvert i_k\rvert]$	23	3	2	12	1	1	12	1
$\Delta(k;\mathbf{i})$	$\Delta_{12,23}$	$\Delta_{1,3}$	$\Delta_{1,2}$	$\Delta_{12,12}$	$\Delta_{1,1}$	$\Delta_{2,1}$	$\Delta_{23,12}$	$\Delta_{3,1}$

Note that these are all the initial minors.

9.4 Cluster algebra structures in $\mathbb{C}[G^{u,v}]$

Let us extend by scalar and consider the upper cluster algebra $\overline{\mathcal{A}}(\mathbf{i})_{\mathbb{C}} := \overline{\mathcal{A}}(\mathbf{i}) \otimes \mathbb{C}$ over the ambient field $\mathcal{F}_{\mathbb{C}} := \mathcal{F} \otimes \mathbb{C}$.

Theorem 9.23. *$F(\mathbf{i})$ is an algebraically independent generating set for the field of rational functions $\mathbb{C}(G^{u,v})$.*

Proof. Note that $|F(\mathbf{i})| = \dim G^{u,v} = r + l(u) + l(v)$ so the dimension is correct. By Theorem 9.5 $\mathbb{C}(G^{u,v}) \simeq \mathbb{C}(H \times C_{\neq 0}^{l(u)+l(v)})$ is generated by $a, t_1, ..., t_m$. These parameters can be expressed explicitly by invertible monomial transforms in the minors of $F(\mathbf{i})$, see Example 9.6 and [FZ]. □

We can state the main result:

Theorem 9.24. *Let $\mathbf{i}$ be a reduced word for $(u, v) \in W \times W$. The isomoprhism of fields $\mathcal{F}_{\mathbb{C}} \longrightarrow \mathbb{C}(G^{u,v})$ defined by*

$$\phi(x_k) \mapsto \Delta(k; \mathbf{i}), \qquad k \in M$$

restricts to an isomorphism of algebras $\overline{\mathcal{A}}(\mathbf{i})_{\mathbb{C}} \longrightarrow \mathbb{C}[G^{u,v}]$

Remark 9.25. *Recently it was shown by Goodearl-Yakimov (2013) that in fact the upper cluster algebra coincides with the cluster algebra: $\overline{\mathcal{A}}(\mathbf{i}) = \mathcal{A}(\mathbf{i})$. It is shown for a large class of* quantized *cluster algebra, in which the coordinate ring of double Bruhat cell is a classical limit of a special case.*

Example 9.26. *Let us calculate the image of ϕ in our Example 9.12. We have the correspondence:*

ϕ	x_{-2}	x_{-1}	x_1	x_2	x_3	x_4	x_5	x_6
	$\Delta_{12,23}$	x_{13}	x_{12}	$\Delta_{12,12}$	x_{11}	x_{21}	$\Delta_{23,12}$	x_{31}

$$\begin{aligned}\phi(x_1') &= \phi\left(\frac{x_{-1}x_2 + x_{-2}x_3}{x_1}\right) \\ &= \frac{x_{13}\Delta_{12,12} + \Delta_{12,23}x_{11}}{x_{12}} \\ &= \Delta_{12,13}\end{aligned}$$

However, not all of them become minors, but they are all regular functions on $G = SL_3(\mathbb{C})$

$$\begin{aligned}\phi(x_2') &= x_{21}\Delta_{13,23} - x_{31}\Delta_{12,23} \\ \phi(x_3') &= x_{22} \\ \phi(x_4') &= \Delta_{13,12}\end{aligned}$$

Remark 9.27. *$\overline{\mathcal{A}}(\mathbf{i})$ is finitely generated (since $\overline{\mathcal{A}}(\mathbf{i})$ is isomorphic to a $\mathbb{Z}$-form of the coordinate ring of a quasi-affine algebraic variety $G^{u,v}$), although it may not have acyclic seed.*

Remark 9.28. *The collection of minors $F(\mathbf{i})$ gives a total positivity criterion in $G^{u,v}$: $x \in G^{u,v}$ is totally positive iff $\Delta(x) > 0$ for every $\Delta \in F(\mathbf{i})$.*

Example 9.29. *From the quiver (red part) in the Figure of Example 9.10, we see that $\mathbb{C}[SL_3^{w_0,w_0}]$ is cluster algebra of type D_4. It has 16 cluster variables and 50 clusters, each consisting of 4 variables.*

- *14 minors (19 minors - det - 4 frozen minors)*
- *2 regular functions: $x_{21}\Delta_{13,23} - x_{31}\Delta_{12,23}$ and $\Delta_{12,13}x_{32} - \Delta_{12,23}x_{31}$*

Example 9.30. *Let $c \in W$ be the Coxeter element with reduced word $(1, 2, 3..., r)$. Then $\mathbb{C}[G^{c,c}]$ is a cluster algebra of type A_1^r, because for $\mathbf{i} = (-1, -2, ..., -r, 1, 2..., r)$, the mutable (red) part of the quiver $\Gamma(\mathbf{i})$ consists of r disconnected vertices, i.e. $B(\mathbf{i})$ is a zero matrix.*

Example 9.31. *Consider $\mathbb{C}[G^{c,c^{-1}}]$, choose $\mathbf{i} = (-1, ..., -r, r, ..., 1)$. Then $B(\mathbf{i})$ is given by*

$$b_{ij} = \begin{cases} -a_{ij} & i < j \\ a_{ij} & i > j \end{cases}$$

Hence the mutable part of $\Gamma(\mathbf{i})$ is a Dynkin graph of G, so that $\mathbb{C}[G^{c,c^{-1}}]$ is a cluster algebra of type G. This gives a geometric construction of cluster algebra of any finite type.

Example 9.32. *The double cell G^{e,w_0} is natually identified with open subset of the base affine space $N_- \setminus G$ given by*

$$\Delta_{\omega_i,\omega_i} \neq 0, \qquad \Delta_{\omega_i,w_0\omega_i} \neq 0, \qquad \forall i = 1, ..., r$$

Then we have

Cartan-Killing type of G	A_2	A_3	A_4	B_2	*other*
Cluster type of $\mathbb{C}[G^{e,w_0}]$	A_1	A_3	D_6	B_2	*infinite*

Idea: show that for other types, the quiver contains the extended Dynkin tree diagram (see Lecture 6), hence it is not 2-finite.

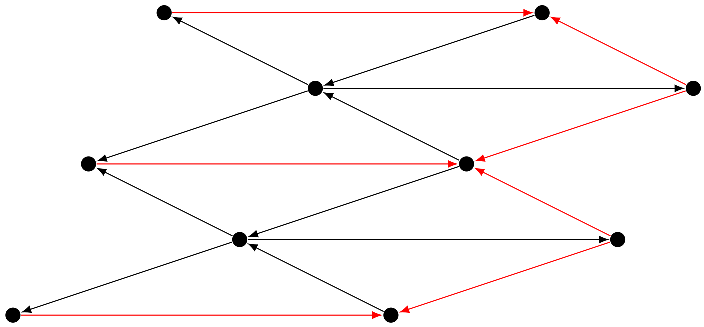

Figure 1: The mutatble part of $\Gamma(\mathbf{i})$ for $\mathbb{C}[G^{e,w_0}]$ in type A_5 for $\mathbf{i} = (1,3,5,2,4,1,3,5,2,4,1,3,5,2,4)$. It contains the subdiagram $E_7^{(1)}$, hence it is of infinite type.

Example 9.33. *For the open double Bruhat cell G^{w_0,w_0}, it is described by*

$$\Delta_{w_0\omega_i,\omega_i} \neq 0, \qquad \Delta_{\omega_i,w_0\omega_i} \neq 0, \qquad \forall i = 1,...,r$$

We have

Cartan-Killing type of G	A_1	A_2	*other*
Cluster type of $\mathbb{C}[G^{w_0,w_0}]$	A_1	D_4	*infinite*

9.5 Proof of Theorem 9.24

We outline some ingredients in the proof of the main theorem.

Lemma 9.34. *(1) The minors $\Delta(k;\mathbf{i})(x) \neq 0$ for $k \notin \mathbf{e}(\mathbf{i})$ and any $x \in G^{u,v}$.*

(2) The map $G^{u,v} \longrightarrow \mathbb{C}^{r+l(u)+l(v)}$ defined by $g \mapsto (\Delta(g))_{\Delta\in F(\mathbf{i})}$ restricts to a biregular isomorphism $U(\mathbf{i}) \longrightarrow \mathbb{C}^{r+l(u)+l(v)}_{\neq 0}$ where

$$U(\mathbf{i}) = \{g \in G^{u,v} : \Delta(g) \neq 0, \qquad \forall\Delta \in F(\mathbf{i})\}$$

Proof. In (1), if $k \notin \mathbf{e}(\mathbf{i})$, then either $u_{\leq k} = e$ or $v_{>k} = e$ (cf. Example 9.22) and $\Delta(k;\mathbf{i})$ turns into $\Delta_{u\omega_i,\omega_i}$ or $\Delta_{\omega_i,v^{-1}\omega_i}$. Hence this is the statement of Theorem 9.19. We can see e.g. that $\Delta_{\omega_i,v^{-1}\omega_i} \neq 0$ follows from Proposition 9.17 and the fact that $B_-vB_-v^{-1} \subset G_0$.

(2) is a restatement of Theorem 9.5, where the parameters $a, t_1, ..., t_N$ can be expressed in terms of minors from $F(\mathbf{i})$. □

Lemma 9.35. *(3) The rational functions $\Delta'(\ell;\mathbf{i}) := \phi(x'_\ell)$ are regular, i.e. belongs to $\mathbb{C}[G^{u,v}]$*

(4) The map $G^{u,v} \longrightarrow \mathbb{C}^{r+l(u)+l(v)}$ *defined by* $g \mapsto (\Delta(g))_{\Delta \in F_\ell(\mathbf{i})}$ *restricts to a biregular isomorphism* $U_\ell(\mathbf{i}) \longrightarrow \mathbb{C}_{\neq 0}^{r+l(u)+l(v)}$ *where*

$$F_\ell(\mathbf{i}) := F(\mathbf{i}) - \{\Delta(\ell;\mathbf{i})\} \cup \{\Delta'(\ell;\mathbf{i}\}$$

and

$$U_\ell(\mathbf{i}) := \{g \in G^{u,v} : \Delta(g) \neq 0, \qquad \forall \Delta \in F_\ell(\mathbf{i})\}$$

Proof. This follows from hard calculations involving identities between the generalized minors developed in [Z, Section 4], see the proof of Lemma 2.12 in [FZ]. Let us outline a special case:

We have the following identity for the generalized minors: if $l(us_i) = l(u)+1$ and $l(vs_i) = l(v)+1$, then

$$\Delta_{u\omega_i, v\omega_i}\Delta_{us_i\omega_i, vs_i\omega_i} - \Delta_{us_i\omega_i, v\omega_i}\Delta_{u\omega_i, vs_i\omega_i} = \prod_{j\neq i} \Delta_{u\omega_j, v\omega_j}^{-c_{ji}}$$

which follows from the case for $u = v = e$.

Recall in our example for $SL_3^{w_0,w_0}$, we have

$$x_2 x_2' = x_{-2}x_3x_5 + x_1x_4$$

which translates to

$$\begin{aligned}
\Delta_{12,12}\Delta' =& \Delta_{12,23}\Delta_{23,12}\Delta_{1,1} + \Delta_{1,2}\Delta_{2,1} \\
=& \Delta_{12,23}\Delta_{23,12}(\Delta_{12,12}\Delta_{13,13} - \Delta_{13,12}\Delta_{12,13}) \\
& + (\Delta_{12,12}\Delta_{13,23} - \Delta_{13,12}\Delta_{12,23})(\Delta_{12,12}\Delta_{23,13} - \Delta_{23,12}\Delta_{12,13}) \\
=& \Delta_{12,12} \det \begin{pmatrix} \Delta_{12,23} & \Delta_{12,13} & \Delta_{12,12} \\ \Delta_{13,23} & \Delta_{13,13} & \Delta_{13,12} \\ 0 & \Delta_{23,13} & \Delta_{23,12} \end{pmatrix}
\end{aligned}$$

Hence Δ' is a regular function on $G^{u,v}$. □

Fact 9.36. *Let* X *be normal variety and* $Y \subset X$ *a subvariety of codimension at least 2. Then any rational function on* X *regular on* $X - Y$ *extends to a regular function on* X.

Lemma 9.37. *Let*

$$U := U(\mathbf{i}) \cup \bigcup_{\ell \in e(\mathbf{i})} U_\ell(\mathbf{i})$$

The complement $G^{u,v} - U$ *has complex codimension at least 2 in* $G^{u,v}$.

Proof. Let $x \in G^{u,v} - U$. Since $x \notin U(\mathbf{i})$, $\Delta(k;\mathbf{i}) = 0$ for some $k \in \mathbf{e}(\mathbf{i})$. Since $x \notin U_k(\mathbf{i})$, either $\Delta(l;\mathbf{i}) = 0$ for some $l \in \mathbf{e}(\mathbf{i})$, or $\Delta'(k;\mathbf{i}) = 0$. Hence $G^{u,v} - U$ is the union of finitely many subvarieties, each given by two distinct irreducible equations. □

Let $\widetilde{\mathbf{x}_\ell} = \widetilde{\mathbf{x}} - \{x_\ell\} \cup \{x'_\ell\}$. We have

$$\overline{\mathcal{A}}(\mathbf{i})_{\mathbb{C}} = \mathbb{C}[\widetilde{\mathbf{x}}^{\pm}] \cap \bigcap_{\ell \in \mathbf{e}(\mathbf{i})} \mathbb{C}[\widetilde{\mathbf{x}}_\ell^{\pm}]$$

Hence we only need to show $\mathcal{C} = \mathbb{C}[G^{u,v}]$ where

$$\mathcal{C} = \phi(\mathbb{C}[\widetilde{\mathbf{x}}^{\pm}]) \cap \bigcap_{\ell \in \mathbf{e}(\mathbf{i})} \phi(\mathbb{C}[\widetilde{\mathbf{x}}_\ell^{\pm}])$$

By Lemma (2) and (4) we have

$$\phi(\mathbb{C}[\widetilde{\mathbf{x}}^{\pm}]) = \mathbb{C}[U(\mathbf{i})], \qquad \phi(\mathbb{C}[\widetilde{\mathbf{x}}_\ell^{\pm}]) = \mathbb{C}[U_\ell(\mathbf{i})]$$

so $\mathcal{C}$ consists of rational functions on $G^{u,v}$ that is regular on U. By Lemma 9.37, these are functions regular on the whole double cell $G^{u,v}$.

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