Displacement Structure of Weighted Pseudoinverses *

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Abstract

Estimates for the rank of $A^{\dagger}_{MN}V - UA^{\dagger}_{MN}$ and more general displacement of $A^{\dagger}_{MN}$ are presented,where $A^{\dagger}_{MN}$ is the weighted pseudoinverse of a matrix A.The results are applied to the close-to-Toeplitz,close-to-Vandermonde and close-to-Cauchy matrices.We extend the results due to G. Heinig and F. Hellinger in 1994.

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1 Introduction

The modern study of structured matrices was largely motivated by [6, 7],in which a basic concept of the displacement rank was introduced.If the rank of a matrix's displacement is small,fast algorithms for the matrix are available. If the UV-displacement of matrix A fulfills a Sylvester equation

$$AU - VA = E,$$

we call it *Sylvester UV-displacement*.If it fulfills a Stein equation

$$A - VAU = E,$$

we call it *Stein UV-displacement*.For example,a Toeplitz matrix T have Stein displacement $T - Z_m^* T Z_n = E$ and the rank of the displacement is 2,so fast algorithms have been constructed.For Sylvester and Stein displacement,if A is invertible, it is easy to show that the

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rank of VU-displacement of A^{-1} is same to the rank of UV-displacement of A. In [2, 3, 18] the VU-displacement of pseudoinverse and group inverse were discussed if A is not invertible.In our paper,we are interested in generalized inverses with as small as possible rank.

We present the representation of the weighted pseudoinverse $A^{\dagger}_{MN}$ of the matrix A.This means we study the so-called displacement $A^{\dagger}_{MN}V - UA^{\dagger}_{MN}$ or $A^{\dagger}_{MN} - UA^{\dagger}_{MN}V$ for structured matrix A.The detail of the weighted pseudoinverse and weighted linear least squares solution can be found in [4, 5, 9, 11-17].

Let $M \in C^{m\times m}$,$N \in C^{n\times n}$ be Hermitian positive definite matrices.We define the weighted inner product in C^m and C^n :

$$(x,y)_M = y^* M x, \quad x,y \in C^m,$$

$$(x,y)_N = y^* N x, \quad x,y \in C^n.$$

So,the weighted conjugate transpose matrix $A^{\sharp} = N^{-1}A^*M$ of $A \in C^{m\times n}$ can be defined by

$$(Ax,y)_M = (x, A^{\sharp}y)_N \quad \forall x \in C^m, y \in C^n.$$

The weighted pseudoinverse of $A \in C^{m\times n}$ is the unique solution $A^{\dagger}_{MN}$[1] of the following four equations:

$$AA^{\dagger}_{MN}A = A, \quad A^{\dagger}_{MN}AA^{\dagger}_{MN} = A^{+}_{MN}, \quad (AA^{\dagger}_{MN})^{\sharp} = AA^{\dagger}_{MN}, \quad (A^{\dagger}_{MN}A)^{\sharp} = A^{\dagger}_{MN}A.$$

From the weighted singular value decomposition (SVD)[10],we know that for any $m \times n$ matrix A with rank r,there exist two weighted unitary matrices $R \in C^{m\times m}$,$S \in C^{n\times n}$ such that

$$A = R \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} S^*, \tag{1}$$

and $R^*MR = I_m$,$S^*N^{-1}S = I_n$,where $\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_r)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0$ are nonzero eigenvalues of $A^{\sharp}A$.The weighted pseudoinverse $A^{\dagger}_{MN}$ has an explicit expression given by:

$$A^{\dagger}_{MN} = N^{-1}S \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} R^*M. \tag{2}$$

We introduce

$$Q_{(MN)} \equiv A^{\dagger}_{MN}A, \quad Q_{(MN)*} \equiv AA^{\dagger}_{MN}, \quad P_{(MN)} \equiv I_n - Q_{(MN)}, \quad P_{(MN)*} \equiv I_m - Q_{(MN)*}.$$

It is easy to show that

$$Im(Q_{(MN)}) = Im(A^{\sharp}) = Im(A^{\dagger}_{MN}), \quad Im(Q_{(MN)*}) = Im(A);$$

$$Im(P_{(MN)}) = Ker(A), \quad Im(P_{(MN)*}) = Ker(A^{\dagger}_{MN}) = Ker(A^{\sharp}).$$

2 Displacement structure for Sylvester displacement

At first,we discuss the Sylvester displacement structure of the weighted pseudoinverse $A^{\dagger}_{MN}$ of $A \in C^{m\times n}$.

Proposition 2.1. Let $A \in C^{m\times n}$,$U \in C^{n\times n}$ and $V \in^{m\times m}$,then

$$A^{\dagger}_{MN}V - UA^{\dagger}_{MN} = A^{\dagger}_{MN}VP_{(MN)*} - P_{(MN)}UA^{\dagger}_{MN} - A^{\dagger}_{MN}(AU - VA)A^{\dagger}_{MN}. \quad (3)$$

Proof. Since

$$A^{\dagger}_{MN}(AU - VA)A^{\dagger}_{MN} = (I - P_{(MN)})UA^{\dagger}_{MN} - A^{\dagger}_{MN}V(I - P_{(MN)*}),$$

we can immediately draw the conclusion. □

Lemma 2.1. The VU-displacement rank of $A^{\dagger}_{MN}$ satisfies the following estimate:

$$rank(A^{\dagger}_{MN}V - UA^{\dagger}_{MN}) \leq rank(Q_{(MN)*}VP_{(MN)*}) + rank(P_{(MN)}UQ_{(MN)}) + rank(AU - VA). \quad (4)$$

Proof.

$$\begin{aligned} rank(A^{\dagger}_{MN}VP_{(MN)*}) &= rank(A^{\dagger}_{MN}VP_{(MN)*})^{\sharp} = dim[P_{(MN)*}V^{\sharp}Im(A^{\dagger}_{MN})^{\sharp}] \\ &= dim[P_{(MN)*}V^{\sharp}Im(Q_{(MN)*})] = rank(Q_{(MN)*}VP_{(MN)*})^{\sharp} \\ &= rank(Q_{(MN)*}VP_{(MN)*}) \end{aligned}$$

$$\begin{aligned} rank(P_{(MN)}UA^{\dagger}_{MN}) &= dim[P_{(MN)}UIm(A^{\dagger}_{MN})] = dim[P_{(MN)}UIm(Q_{(MN)})] \\ &= rank(P_{(MN)}UQ_{(MN)}) \end{aligned}$$

Taking these two into account,we obtain (4). □

Proposition 2.2. Let $A \in C^{m\times n}$,$U \in C^{n\times n}$ and $V \in^{m\times m}$,then

$$rank(P_{(MN)}UQ_{(MN)}) + rank(Q_{(MN)*}VP_{(MN)*}) \leq rank(AU^{\sharp} - V^{\sharp}A), \quad (5)$$

where $U^{\sharp} = N^{-1}U^{*}N$ and $V^{\sharp} = M^{-1}V^{*}M$.

Proof. We set $F \equiv AU^{\sharp} - V^{\sharp}A$.Let

$$A = R\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}S^{*}$$

be the weighted SVD of A.Partition

$$R^{*}MV^{\sharp}R = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad S^{*}U^{\sharp}N^{-1}S = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

where $U_{11}, V_{11} \in C^{r\times r}$ and $r = rank(A)$.Therefore,

$$R^* MFN^{-1}S = \begin{pmatrix} \Sigma U_{11} - V_{11}\Sigma & \Sigma U_{12} \\ V_{21}\Sigma & 0 \end{pmatrix}.$$

Since

$$Q_{(MN)}U^\sharp P_{(MN)} = N^{-1}S \begin{pmatrix} 0 & U_{12} \\ 0 & 0 \end{pmatrix} S^*$$

and

$$P_{(MN)*}V^\sharp Q_{(MN)*} = R \begin{pmatrix} 0 & 0 \\ V_{21} & 0 \end{pmatrix} R^* M,$$

it follows from [8] that

$$\begin{aligned} rank(F) &= rank(R^* MFN^{-1}S) \\ &\geq rank(\Sigma U_{12}) + rank(V_{21}\Sigma) \\ &= rank(U_{12}) + rank(V_{21}) \\ &= rank(Q_{(MN)}U^\sharp P_{(MN)}) + rank(P_{(MN)*}V^\sharp Q_{(MN)*}) \\ &= rank(P_{(MN)}UQ_{(MN)}) + rank(Q_{(MN)*}VP_{(MN)*}). \end{aligned}$$

□

From Proposition 2.1,2.2 and Lemma 2.1,we conclude that

Theorem 2.1. Let $A \in C^{m\times n}$ and $A^\dagger_{MN}$ its weighted pseudoinverse.Then

$$rank(A^\dagger_{MN}V - UA^\dagger_{MN}) \leq rank(AU - VA) + rank(AU^\sharp - V^\sharp A). \tag{6}$$

□

Corollary 2.1. If U, V are both weighted self-adjoint $U = U^\sharp, V = V^\sharp$ or weighted unitary $U^{-1} = U^\sharp, V^{-1} = V^\sharp$,then

$$rank(A^\dagger_{MN}V - UA^\dagger_{MN}) \leq 2\ rank(AU - VA). \tag{7}$$

□

3 Displacement structure for generalized displacement

In order to generalize Theorem 2.1 we introduce a generalized displacement concept[2].Let $a = [a_{ij}]_0^1$ denote a nonsingular 2×2 matrix.We associate a with the polynomial in two variables

$$a(\lambda, \mu) = \sum_{i,j=0}^{1} a_{ij}\lambda^i \mu^j$$

and the linear fractional function

$$f_a(\lambda) = \frac{a_{10} + a_{11}\lambda}{a_{00} + a_{01}\lambda}.$$

For any fixed $U \in C^{n\times n}$ and $V \in C^{m\times m}$,the generalized (a, U, V) displacement of $A \in C^{m\times n}$ generated by $a(\lambda, \mu)$ is defined by

$$a(V, U)A = \sum_{i,j=0}^{1} a_{ij} V^i A U^j.$$

If

$$a = d \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we just get Sylvester displacement that we have discussed in Section 2.If

$$a = d \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we get Stein displacement.

Lemma 3.1.[2] Let $a = [a_{ij}]_0^1$,$b = [b_{ij}]_0^1$,$c = [c_{ij}]_0^1$,$d = [d_{ij}]_0^1$ be nonsingular 2×2 matrices such that

$$a = b^T dc, \tag{8}$$

then

$$(b_{00} + b_{01}\lambda)^{-1} a(\lambda, \mu)(c_{00} + c_{01}\mu)^{-1} = d(f_b(\lambda), f_c(\mu)). \tag{9}$$

□

Lemma 3.2.[2] Let $d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,then there exist 2×2 matrices b,c such that (8) holds and $b_{00} + b_{01}V$ and $c_{00} + c_{01}U$ are invertible. □

Taken Lemma 3.1 and Lemma 3.2 together,we obtain the following

Proposition 3.1.[2] Let b and c be matrices satisfying the conditions in Lemma 3.2,then for $A \in C^{m\times n}$,

$$a(V, U)A = (b_{00} + b_{01}V)[Af_c(U) - f_b(V)A](c_{00} + c_{01}U).$$

□

The following is very important to generalize Theorem 2.1 for general (a, U, V) displacement.

Proposition 3.2.

(a) If $\phi = [\phi_{ij}]_0^1$ is nonsingular and $\phi_{00} + \phi_{01}U$ is invertible,then

$$rank(P_{(MN)} U Q_{(MN)}) = rank(P_{(MN)} \tilde{U} Q_{(MN)}),$$

where $\tilde{U} \equiv f_\phi(U)$.

(b) If $\psi = [\psi_{ij}]_0^1$ is nonsingular and $\psi_{00} + \psi_{01}V$,then

$$rank(Q_{(MN)*}VP_{(MN)*}) = rank(Q_{(MN)*}\tilde{V}P_{(MN)*}),$$

where $\tilde{V} \equiv f_\psi(V)$.

Proof. We define

$$\mathcal{S} = Ker(A) \cap Ker(AU^\sharp) \quad , \quad \mathcal{S}_1 = Ker(A) \ominus \mathcal{S}.$$

We show that $Q_{(MN)}U^\sharp$ is one-to-one on $\mathcal{S}_1$.If $Q_{(MN)}U^\sharp x = 0$ and $x \in \mathcal{S}_1$,then $U^\sharp x \in Ker(Q_{(MN)}) = Ker(A)$.That means $AU^\sharp x = 0$.Noting that $x \in Ker(A)$,we conclude $x \in \mathcal{S}$.Thus $x = 0$.

Furthermore,$Q_{(MN)}U^\sharp x = 0$ for all $x \in \mathcal{S}$.Hence

$$rank(P_{(MN)}UQ_{(MN)}) = rank(Q_{(MN)}U^\sharp P_{(MN)}) = dim(\mathcal{S}_1). \tag{10}$$

Analogously we define

$$\tilde{\mathcal{S}} = Ker(A) \cap Ker(A\tilde{U}^\sharp) \quad , \quad \tilde{\mathcal{S}}_1 = Ker(A) \ominus \tilde{\mathcal{S}},$$

and we will get

$$rank(P_{(MN)}\tilde{U}Q_{(MN)}) = rank(Q_{(MN)}\tilde{U}^\sharp P_{(MN)}) = dim(\tilde{\mathcal{S}}_1). \tag{11}$$

Now we show that the invertible matrix $\overline{\phi}_{00} + \overline{\phi}_{01}U^\sharp$ bijectively maps $\mathcal{S}$ onto $\tilde{\mathcal{S}}$.Suppose that $x \in \mathcal{S}$.Then $x, U^\sharp x \in Ker(A)$.Hence $y \equiv (\overline{\phi}_{10} + \overline{\phi}_{11}U^\sharp)x$ and $z \equiv (\overline{\phi}_{00} + \overline{\phi}_{01}U^\sharp)x$ are all contained in $Ker(A)$.Thus $y = \tilde{U}^\sharp z$ and we conclude that $z, \tilde{U}^\sharp z \in Ker(A)$,which implies $z \in \tilde{\mathcal{S}}$.Convesely,with the same arguments we get $(\overline{\phi}_{00} + \overline{\phi}_{01}U^\sharp)^{-1}z \in \mathcal{S}$ for all $z \in \tilde{\mathcal{S}}$.

This implies

$$dim(\mathcal{S}_1) = dim[Ker(A)] - dim(\mathcal{S}) = dim[Ker(A)] - dim(\tilde{\mathcal{S}}) = dim(\tilde{\mathcal{S}}_1).$$

According to (10) and (11),we get assertion (a).

Assertion (b) is proved analogously. □

Now we can generalize Theorem 2.1 for general (a, U, V) displacement.

Theorem 3.1. Let a,b be 2×2 nonsingular matrices,then

$$rank[a(U, V)A_{MN}^\dagger)] \leq rank[a^T(V, U)A] + rank[b(V^\sharp, U^\sharp)A]. \tag{12}$$

Proof. According to Lemma 3.2 there exist 2×2 matrices w,x,y,z such that $w_{00}+w_{01}U$, $x_{00}+x_{01}V$, $y_{00}+y_{01}U$, $z_{00}+z_{01}V$ are invertible and

$$a = w^T dz,$$

$$b = x^* d\overline{y}.$$

Hence,

$$\begin{aligned} & rank[a(U,V)A^{\dagger}_{MN}] - rank[a^T(V,U)A] \\ = \;& rank[f_w(U)A^{\dagger}_{MN} - A^{\dagger}_{MN} f_z(V)] - rank[f_z(V)A - Af_w(U)] \\ \leq \;& rank[P_{(MN)} f_w(U) Q_{(MN)}] + rank[Q_{(MN)*} f_z(V) P_{(MN)*}] \\ = \;& rank[P_{(MN)} f_y(U) Q_{(MN)}] + rank[Q_{(MN)*} f_x(V) P_{(MN)*}] \\ \leq \;& rank[f_{\overline{x}}(V^{\sharp})A - Af_{\overline{y}}(U^{\sharp})] \\ = \;& rank[b(V^{\sharp},U^{\sharp})A]. \end{aligned}$$

□

Corollary 3.1. If U,V are weighted unitary or weighted self-adjoint matrix,then

$$rank[a(U,V)A^{\dagger}_{MN}] \leq 2\ rank[a^T(V,U)A]. \tag{13}$$

Proof. Let

$$b = \begin{cases} a^T & \text{if } U^{\sharp} = U, V^{\sharp} = V, \\ ia^T & \text{if } U^{\sharp} = U, V^{\sharp} = V^{-1}, \\ a^T i & \text{if } U^{\sharp} = U^{-1}, V^{\sharp} = V, \\ ia^T i & \text{if } U^{\sharp} = U^{-1}, V^{\sharp} = V^{-1}, \end{cases}$$

where i denote the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.We immediately obtain (13) from Theorem 3.1. □

4 Computation of the displacement

For practical purpose it is important to know not only the displacement rank of $A^{\dagger}_{MN}$ but the explicit form of the displacement.For simplicity we restrict our explantation to the case of Sylvester displacement and to the case of a matrix A.

Our starting point is (3).

(1) If we know the full-rank decomposition (For many structured matrices,it is easy to get the decomposition.)

$$AU - VA = GF^* = \sum_{i=1}^{r} g_i f_i^*,$$

we only need to compute the weighted least squares solutions[1, 15] $A^{\dagger}_{MN}g_i$ and $f_i^* A^{\dagger}_{MN}$ to get

$$A^{\dagger}_{MN}(AU - VA)A^{\dagger}_{MN}.$$

(2) For the purpose to get $A^{\dagger}_{MN}VP_{(MN)*}$,we start with another full-rank decomposition

$$A^{\sharp}V - UA^{\sharp} = KL^*.$$

We denote $C \equiv \begin{bmatrix} A^*M^{1/2} \\ L^*M^{-1/2} \end{bmatrix}$.It is obvious that $Ker(C) \subseteq M^{-1/2}Ker(A^*)$,so we can find an orthonormal system of vectors $w_1, \cdots, w_p$ forming a basis of the orthogonal complement of $Ker(C)$ in $M^{-1/2}Ker(A^*)$.If we introduce the matrix $W = [w_1, \cdots, w_p]$,then we have

$$M^{-1/2}Ker(A^*) = Ker(C) \oplus Im(W).$$

Proposition 4.1. Let $R \equiv M^{-1/2}WW^*M^{1/2}$,then

$$RP_{(MN)*} = R. \tag{14}$$

Proof. It is obvious that

$$Im(RQ_{(MN)*}) = RIm(Q_{(MN)*}) = RIm(A) = Im(RA) = Im(M^{-1/2}WW^*M^{1/2}A).$$

Noting that $Im(W) \subseteq M^{-1/2}Ker(A^*)$,we get

$$A^*M^{1/2}W = 0\ .$$

Hence,$W^*M^{1/2}A = 0$.So,we conclude $RQ_{(MN)*} = 0$.Taking $P_{(MN)*} = I_m - Q_{(MN)*}$ into account,we obtain (14). □

Proposition 4.2. Let R be defined as Proposition 4.1,then

$$A^{\dagger}_{MN}V(I_m - R)P_{(MN)*} = 0. \tag{15}$$

Proof. In view of $Im(P_{(MN)*}) = Ker(A^{\sharp})$,one can easily check

$$\begin{aligned} Im[A^{\dagger}_{MN}V(I_m - R)P_{(MN)*}] &= A^{\dagger}_{MN}VM^{-1/2}(I_m - WW^*)M^{1/2}Ker(A^{\sharp}) \\ &= A^{\dagger}_{MN}VM^{-1/2}(I_m - WW^*)M^{-1/2}Ker(A^*). \end{aligned}$$

For all $x \in M^{-1/2}Ker(A^*)$,there exists a unique decomposition $x = y + z$ such that $y \in Ker(C)$ and $z \in Im(W)$. Noting WW^* is an orthogonal projection onto $Im(W)$,we get $(I_m - WW^*)(y + z) = y \in Ker(C)$.Hence

$$Im[A^{\dagger}_{MN}V(I_m - R)P_{(MN)*}] \subseteq A^{\dagger}_{MN}VM^{-1/2}Ker(C). \tag{16}$$

Now we show $A^{\dagger}_{MN}VM^{1/2}Ker(C)=0$.Surpose $x\in Ker(C)$.Then

$$A^*M^{1/2}x=0 \ \text{ and } \ L^*M^{-1/2}x=0.$$

Hence,

$$A^{\sharp}VM^{-1/2}x-UA^{\sharp}M^{-1/2}x=KL^*M^{-1/2}x=0.$$

Thus,

$$A^{\sharp}VM^{-1/2}x=UA^{\sharp}M^{-1/2}x=UN^{-1}A^*M^{1/2}x=0.$$

So,we have $VM^{-1/2}x\in Ker(A^{\sharp})=Ker(A^{\dagger}_{MN})$.This means $A^{\dagger}_{MN}VM^{-1/2}Ker(C)=0$.

Noting (16),we obtain (15). □

According to Proposition 4.1 and 4.2,we have

$$A^{\dagger}_{MN}VP_{(MN)*}=A^{\dagger}_{MN}VM^{-1/2}WW^*M^{1/2}.$$

(3) We proceed analogously for $P_{(MN)}UA^{\dagger}_{MN}$. Let $C_*\equiv\begin{bmatrix}AN^{-1/2}\\K^*N^{1/2}\end{bmatrix}$ and $S\equiv N^{-1/2}ZZ^*N^{1/2}$, where $Z=[z_1,\cdots,z_q]$ and $z_1,\cdots,z_q$ is an orthonormal basis of the orthogonal complement of $Ker(C_*)$ in $Ker(AN^{-1/2})$.The result obtained is

$$P_{(MN)}UA^{\dagger}_{MN}=SUA^{\dagger}_{MN}=N^{-1/2}ZZ^*N^{1/2}UA^{\dagger}_{MN}.$$

In order to compute the displacement,one have to find $2r$ weighted least squares solutions $A^{\dagger}_{MN}g_i$ and $f_i^*A^{\dagger}_{MN}$,where $r=rank(AU-VA)$,and $p+q$ weighted least squares solutions $A^{\dagger}_{MN}VM^{-1/2}w_i$ $(i=1,\cdots,p)$ and $z_j^*N^{1/2}UA^{\dagger}_{MN}$ $(j=1,\cdots,q)$,where $p+q\leq rank(A^{\sharp}V-UA^{\sharp})$.

5 Full rank matrices

In this section we consider a special case that A has full rank.If this condition is fulfilled,then $A^{\dagger}_{MN}$ has an explicit form given by $A^{\dagger}_{MN}=A^{\sharp}(AA^{\sharp})^{-1}$ or $A^{\dagger}_{MN}=(A^{\sharp}A)^{-1}A^{\sharp}$.

We show that under the assumption made above one can find a more general estimate for the displacement rank.In fact,in the case under consideration the $U^{\sharp}V^{\sharp}$-displacement can be displaced by the $U^{\sharp}W^{\sharp}$-displacement for arbitrary $W\in C^{m\times m}$,or by the $W^{\sharp}V^{\sharp}$-displacement for arbitrary $W\in C^{n\times n}$.This is important for a series of applications.

We will start with Sylvester displacement as we have done in Section 2.Then we generalize it to the generalized displacement.

Proposition 5.1. Let A,U,V,$P_{(MN)}$,$P_{(MN)*}$ be defined as before and $W_1 \in C^{n\times n}$,$W_2 \in C^{m\times m}$ are arbitrary,then

$$\begin{aligned} A^{\dagger}_{MN}V - UA^{\dagger}_{MN} &= (A^{\sharp}A + P_{(MN)})^{-1}(A^{\sharp}V - W_1A^{\sharp})P_{(MN)*} \\ &+ P_{(MN)}(A^{\sharp}W_2 - UA^{\sharp})(AA^{\sharp} + P_{(MN)*})^{-1} - A^{\dagger}_{MN}(AU - VA)A^{\dagger}_{MN}. \end{aligned}$$

Proof. By the weighted SVD,we have $P_{(MN)}A^{\sharp} = 0$ and $A^{\sharp}P_{(MN)*} = 0$.Hence,

$$P_{(MN)}A^{\sharp}W_2(AA^{\sharp} + P_{(MN)}*)^{-1} = 0 \quad \text{and} \quad (A^{\sharp}A + P_{(MN)})^{-1}W_1A^{\sharp}P_{(MN)*} = 0.$$

According to the weighted SVD,one can easily check that

$$A^{\dagger}_{MN} = (A^{\sharp}A + P_{(MN)})^{-1}A^{\sharp} = A^{\sharp}(AA^{\sharp} + P_{(MN)*})^{-1}.$$

Taking this into account and noting (3),we get the result. □

We can immediately get the following two theorems through Proposition 5.1.

Theorem 5.1. Let $A \in C^{m\times n}$ be of row full rank and $m < n$,then for any arbitrary $W \in C^{m\times m}$,

$$rank(A^{\dagger}_{MN}V - UA^{\dagger}_{MN}) \le rank(AU - VA) + rank(AU^{\sharp} - W^{\sharp}A). \tag{17}$$

□

Theorem 5.2. Let $A \in C^{m\times n}$ be of column full rank and $m > n$,then for any arbitrary $W \in C^{n\times n}$,

$$rank(A^{\dagger}_{MN}V - UA^{\dagger}_{MN}) \le rank(AU - VA) + rank(AW^{\sharp} - V^{\sharp}A). \tag{18}$$

□

Now we turn to the generalized displacement.

Theorem 5.3. Let $A \in C^{m\times n}$ be of full rank and $m < n$,let a,b be nonsingular 2×2 matrices,then for any arbitrary $W \in C^{m\times m}$,

$$rank[a(U, V)A^{\dagger}_{MN}] \le rank[a^T(V, U)A] + rank[b(W^{\sharp}, U^{\sharp})A]. \tag{19}$$

Proof. Under the assumptions there exist 2×2 matrices w and z such that $a = w^T dz$ and the matrices $w_{00} + w_{01}U$ and $z_{00} + z_{01}V$ are invertible. Furthermore,

$$a(U, V)A^{\dagger}_{MN} = (w_{00} + w_{01}U)[A^{\dagger}_{MN}f_z(V) - f_w(U)A^{\dagger}_{MN}](z_{00} + z_{01}V)$$

together with

$$A^{\dagger}_{MN}f_z(V) - f_w(U)A^{\dagger}_{MN} = -P_{(MN)}f_w(U)Q_{(MN)} - A^{\dagger}_{MN}[Af_w(U) - f_z(V)A]A^{\dagger}_{MN}$$

implies

$$rank[a(U,V)A^{\dagger}_{MN}] \leq rank[P_{(MN)}f_w(U)Q_{(MN)}] + rank[Af_w(U) - f_z(V)A].$$

Noting $a^T = -z^T dw$,we get

$$rank[Af_w(U) - f_z(V)A] = rank[a^T(V,U)A].$$

Under the assumptions there also exist 2×2 matrices x and y such that $a = x^T dy$ and the matrices $x_{00} + x_{01}W$ and $y_{00} + y_{01}U$ are invertible.Then

$$\begin{aligned} rank[P_{(MN)}f_w(U)Q_{(MN)}] &= rank[P_{(MN)}f_y(U)Q_{(MN)}] \\ &= rank[P_{(MN)}f_y(U)Q_{(MN)}] + rank[Q_{(MN)*}f_x(W)P_{(MN)*}] \\ &\leq rank[Af_{\bar{y}}(U^{\sharp}) - f_{\bar{x}}(W^{\sharp})A] \\ &= rank[b(W^{\sharp},U^{\sharp})A]. \end{aligned}$$

□

The following theorem can be proved analogously.

Theorem 5.4. Let $A \in C^{m\times n}$ be of full column rank and $m > n$,let a,b be nonsingular 2×2 matrices,then for any arbitrary $W \in C^{n\times n}$,

$$rank[a(U,V)A^{\dagger}_{MN}] \leq rank[a^T(V,U)A] + rank[b(V^{\sharp},W^{\sharp})A]. \tag{20}$$

□

6 Applications

In this section we apply the theorems proved before to many classical structured matrices,including Toeplitz,Hankel,Cauchy and Vandermonde matrices.

6.1 Close-to-Toeplitz matrices

Close-to-Toeplitz matrices are a class of matrices whose UV-displacement ranks are small compared with the sizes of the matrices for U and V being (forward or backward) (block) shifts,including Toeplitz,Hankel matrices,more geneal block matrices with Toeplitz or Hankel blocks,and sums,products,and inverses of these matrices.

We consider the case

$$U = Z_n \equiv \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in C^{n\times n} \ , \quad V = Z_m^*$$

and let

$$r_+ \equiv rank(A - Z_m^* A Z_n) \quad , r_- \equiv rank(A - Z_m A Z_n^*).$$

Choosing

$$a = b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in Theorem 3.1 and noting

$$rank(A - (Z_m^*)^\sharp A Z_n^\sharp) = rank(MAN^{-1} - Z_m MAN^{-1} Z_n^*),$$

we obtain the following

Theorem 6.1.1. Let $r_{MAN^{-1}} = rank(MAN^{-1} - Z_m MAN^{-1} Z_n^*)$,then

$$rank(A_{MN}^\dagger - Z_n A_{MN}^\dagger Z_m^*) \leq r_+ + r_{MAN^{-1}} \ .$$

□

If the estimate of Theorem 6.1.1 is small for a close-to-matrices A,then it lead to the the famous representation formula of Gohberg-Semencul type of $A_{MN}^\dagger$:

$$A_{MN}^\dagger = \sum_{k=1}^{r} L_k U_k \ ,$$

where r is the displacement rank of $A_{MN}^\dagger$,i.e.,$r_+ + r_{MAN^{-1}}$ here.

The importance of the representations consists the fact that with their help weighted least squares solutions $A_{MN}^\dagger$ for a close-to-Toeplitz matrix A can be computed with the complexity $O((m+n)log(m+n))$ if the FFT is applied.

Corollary 6.1.1.

$$rank(A^\dagger - Z_n A^\dagger Z_m^*) \leq r_+ + r_-.$$

□

Corollary 6.1.2. Let $r_M = rank(M - Z_m M Z_m^*)$,$r_N = rank(N - Z_n^* N Z_n)$,then

$$rank(A_{MN}^\dagger - Z_n A_{MN}^\dagger Z_m^*) \leq 2r_+ + r_M + r_N.$$

Proof. By [6],we have

$$rank(N - Z_n^* N Z_n) = rank(N^{-1} - Z_n N^{-1} Z_n^*).$$

We immediately obtain the corollary by the following

$$\begin{aligned} & MAN^{-1} - Z_m MAN^{-1} Z_n^* \\ = \ & (M - Z_m M Z_m^*) AN^{-1} + Z_m M Z_m^* A (N^{-1} - Z_n N^{-1} Z_n^*) - Z_m M (A - Z_m^* A Z_n) N^{-1} Z_n^*. \end{aligned}$$

□

However,the estimate in Corollary 6.1.2 is not always small,but if we choose the weight matrices M,N such that r_M,r_N is very small compared to the size of A,then the rank of the VU-displacement of $A^{\dagger}_{MN}$ is also very small.For example,let M,N be Hermite positive definite Toeplitz matrices,then the displacement rank of the weighted pseudoinverse for a Toeplitz matrix is less than or equal to 8 through Corollary 6.1.2.

6.2 Close-to-Vandermonde matrices

Let $D(c) = diag(c_1, \cdots, c_m)$.It is well known that the displacement rank $r = rank[Van_n(c)Z_n - D(c)Van_n(c)]$ of a Vandermonde matrix $Van_n(c) = [c_i^{j-1}]_{i=1,j=1}^{m,n}$ is equal to one,except for the trivial case $r = 0$.Hence,an $m \times n$ matrix is said to be close-to-Vandermonde if,for certain $c \in C^m$,the displacement rank $rank[AZ_n - D(c)A]$ is small compared with m and n.

We denote

$$r \equiv rank[AZ_n - D(c)A] \ , \quad r' \equiv rank[A - D(c)AZ_n^*].$$

It is easily to check that a close-to-Vandermonde matrix admits a representation

$$A = \sum_{i=1}^{r} D_i Van_n(c) T_i + D_0 Van_n(c), \tag{21}$$

where D_i are diagonal matrices and T_i are upper triangular Toeplitz matrices with zeros at the main diagonal.Note that the matrices D_i and T_i can be found via the full-rank decomposition of $AZ_n - D(c)A$,and D_0 is related to the first column of A.

With the representation (21),we can show $r' \leq r+1$ and in particular,if A is Vandermonde matrix,$r' = r = 1$.

Theorem 6.2.1. Let $r_{MAN^{-1}} = MAN^{-1} - D(c)MAN^{-1}Z_n^*$.Suppose that $c_i \in R$ or $|c_i| = 1$ for all $i = 1, \cdots, m$,then

$$rank[A^{\dagger}_{MN}D(c) - Z_n A^{\dagger}_{MN}] \leq r + r_{MAN^{-1}}.$$

Proof. Set $b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ if $c_i \in R$ and set $b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ if $|c_i| = 1$ in Theorem 3.1.Since

$$rank[AZ_n^{\sharp} - D(c)^{\sharp}A] = rank[MAN^{-1}Z_n^* - D(c)^*MAN^{-1}]$$

and

$$rank[A - D(c)^{\sharp}AZ_n^{\sharp}] = rank[MAN^{-1} - D(c)^*MAN^{-1}Z_n^*],$$

we immediately get the result. □

We give the following corollary.

Corollary 6.2.1. Suppose that $c_i \in R$ or $|c_i| = 1$ for all $i = 1, \cdots, m$,then

$$rank[A^{\dagger}D(c) - Z_n A^{\dagger}] \leq r + r' \leq 2r + 1.$$

□

Theorem 6.2.2. Let $r_N = rank(N - Z_n^* N Z_n)$,$r_M = rank[M - D(c)MD(c)^*]$.Suppose that $|c_i| = 1$ for all $i = 1, \cdots, m$,then

$$rank[A^{\dagger}_{MN}D(c) - Z_n A^{\dagger}_{MN}] \leq 2r + r_M + r_N.$$

Proof. Since

$$\begin{aligned} & MAN^{-1} - D(c)MAN^{-1}Z_n^* \\ = \;& (M - D(c)MD(c)^*)AN^{-1} + D(c)MD(c)^*A(N^{-1} - Z_n N^{-1} Z_n^*) \\ & -D(c)M(A - D(c)^*AZ_n)N^{-1}Z_n^*, \end{aligned}$$

noting $D(c)^* = D(c)^{-1}$ and by [6],we have

$$rank(N - Z_n^* N Z_n) = rank(N^{-1} - Z_n N^{-1} Z_n^*),$$

we obtain the theorem. □

Theorem 6.2.3. Let $r_N^{'} = rank(NZ_n - Z_n^* N)$,$r_M^{'} = rank[MD(c) - D(c)M]$.Suppose that $c_i \in R$ for all $i = 1, \cdots, m$,then

$$rank[A^{\dagger}_{MN}D(c) - Z_n A^{\dagger}_{MN}] \leq 2r + r_M^{'} + r_N^{'}.$$

Proof. Set $b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in Theorem 3.1.Therefore,

$$\begin{aligned} rank[A^{\dagger}_{MN}D(c) - Z_n A^{\dagger}_{MN}] & \leq r + rank[AZ_n^{\sharp} - D(c)^{\sharp}A] \\ & = r + rank[MAN^{-1}Z_n^* - D(c)^*MAN^{-1}]. \end{aligned}$$

Since

$$\begin{aligned} & MAN^{-1}Z_n^* - D(c)^*MAN^{-1} \\ = \;& (MD(c) - D(c)^*M)AN^{-1} + MA(N^{-1}Z_n^* - Z_n N^{-1}) + M(AZ_n - D(c)A)N^{-1}, \end{aligned}$$

noting $D(c)^* = D(c)$ and

$$rank(NZ_n - Z_n^* N) = rank(N^{-1}Z_n^* - Z_n N^{-1}),$$

we obtain the theorem. □

6.3 Generalized Cauchy matrices

Let U,V be diagonal matrices,

$$U \equiv D(d) = diag(d_1, \cdots, d_n) \quad , \quad V \equiv D(c) = diag(c_1, \cdots, c_m).$$

A matrix A is said to be a generalized Cauchy matrix if for certain c and d,$rank[AD(d) - D(c)A]$ is small compared with m and n.In case $c_i \neq d_j$ for all i and j,it has an explicit form

$$A = \left[\frac{f_i^* g_j}{c_i - d_j} \right]_{i=1,j=1}^{m,n} , \tag{22}$$

where $f_i, g_j \in C^r$ and $r = rank[AD(d) - D(c)A]$.

In particular,if $r = 1$,$f_1 = g_1 = 1$, A is classical Cauchy Matrix.If $f_1 = a = (a_1, \cdots, a_m)^*, f_2 = (-1, \cdots, -1)^*, g_1 = (1, \cdots, 1)^*, g_2 = b = (b_1, \cdots, b_n)^*$,$A$ is Loewner matrix,which has the form

$$A = \left[\frac{a_i - b_j}{c_i - d_j} \right]_{i=1,j=1}^{m,n} .$$

We assume that $c_i \in R$ or $|c_i| = 1$ for all i,and the same for d_j.In case $c \in R^m$,we have $D(c)^* = D(c)$;in case $|c_i| = 1$,we have $D(c)^* = D(c)^{-1}$.

Theorem 6.3.1. Let $r_{MAN^{-1}} = rank[MAN^{-1}D(d) - D(c)MAN^{-1}]$.Suppose that $c_i \in R$ or $|c_i| = 1$ for all $i = 1, \cdots, m$ and $d_j \in R$ or $|d_j| = 1$ for all $j = 1, \cdots, n$.Then

$$rank[A_{MN}^{\dagger} D(c) - D(d) A_{MN}^{\dagger}] \leq r + r_{MAN^{-1}} ,$$

where $r = rank[AD(d) - D(c)A]$.

Proof. Set

$$b = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \text{if } c_i \in R, d_i \in R \text{ or } |c_i| = 1, |d_i| = 1 \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \text{if } c_i \in R, |d_i| = 1 \text{ or } |c_i| = 1, d_i \in R \end{cases}$$

in Theorem 3.1,we immediately obtain the theorem. □

Corollary 6.3.1. If $c_i \in R$ or $|c_i| = 1$ for all i,and the same for d_j.Then

$$rank[A^{\dagger} D(c) - D(d) A^{\dagger}] \leq 2r,$$

where $r = rank[AD(d) - D(c)A]$. □

Corollary 6.3.2. Let $r_M = rank[MD(c) - D(c)M]$,$r_N = rank[ND(d) - D(d)N]$.If $c_i \in R$ or $|c_i| = 1$ for all i,and the same for d_j,then

$$rank[A_{MN}^{\dagger} D(c) - D(d) A_{MN}^{\dagger}] \leq 2r + r_M + r_N,$$

where $r = rank[AD(d) - D(c)A]$.

Proof. Since

$$\begin{aligned} & MAN^{-1}D(d) - D(c)MAN^{-1} \\ = \ & MA(N^{-1}D(d) - D(d)N^{-1}) + M(AD(d) - D(c)A)N^{-1} - (D(c)M - MD(c))AN^{-1} \end{aligned}$$

and

$$rank[N^{-1}D(d) - D(d)N^{-1}] = rank[ND(d) - D(d)N],$$

we obtain

$$r_{MAN^{-1}} \leq r + r_N + r_M.$$

□

With Theorem 6.3.1,we conclude that $A^{\dagger}_{MN}$ has the form (22).It indicates that one can construct a fast algorithm to compute the weighted least squares solution $x = A^{\dagger}_{MN}y$ for a generalized Cauchy system if $r + r_{MAN^{-1}}$ is small.

6.4 Upper bound for full rank matrices

In the above subsections,we give displacement estimates of A by the displacement of weight matrices M and N.In this subsection,we give an upper bound of the displacement ranks independent of the displacement of weight matrices.At first,we consider the case that A has full column rank and the Sylvester displacement.

Our start point is (3).Assume $m > n$.Because A has full column rank,so (3) can be written into

$$A^{\dagger}_{MN}V - UA^{\dagger}_{MN} = A^{\dagger}_{MN}VP_{(MN)*} - A^{\dagger}_{MN}(AU - VA)A^{\dagger}_{MN}.$$

Set

$$K \equiv A^{\dagger}_{MN}VP_{(MN)*}.$$

We consider the system $K^{\sharp}x = 0$.This system can be written into the form

$$P_{(MN)*}V^{\sharp}(A^{\dagger}_{MN})^{\sharp}x = 0.$$

Since $Im[(A^{\dagger}_{MN})^{\sharp}] = Im(A)$,we get

$$rank(P_{(MN)*}V^{\sharp}A) = rank[P_{(MN)*}V^{\sharp}(A^{\dagger}_{MN})^{\sharp}].$$

Hence,the dimension of the solution space of $K^{\sharp}x = 0$ is equivalent to the dimension of the solution space of the following

$$P_{(MN)*}V^{\sharp}Ax = 0. \tag{23}$$

Now we consider another equation

$$(V^{\sharp}A - AW)x = 0 \quad , \quad W \in C^{n\times n}. \tag{24}$$

If x is a solution of (24),then

$$V^{\sharp}Ax = AWx \in Im(A) = Ker(P_{(MN)*}).$$

It means the solution space of (24) is a subset of solution space of (23).Therefore,if we give a lower bound of the solution space's dimension of (24),let it be d,then $n-d$ is the upper bound of $rank(K)$.In fact,if we set $W = A^{\dagger}V^{\sharp}A$,(24) is changed into

$$(I_m - AA^{\dagger})V^{\sharp}Ax = R\begin{bmatrix} 0 & 0 \\ 0 & I_{m-n} \end{bmatrix} R^{*}V^{\sharp}Ax = 0.$$

Then we obtain $d = n-(m-n) = 2n-m$,therefore,

$$rank(K) \leq n - (2n-m) = m-n.$$

We generalize it to the generalized displacement $a(U,V)A^{\dagger}_{MN}$.By Lemma 3.2,there exist nonsingular 2×2 matrices w and z such that $a = w^T dz$ and $w_{00}+w_{01}U$ and $z_{00}+z_{01}V$ are invertible.Then

$$a(U,V)A^{\dagger}_{MN} = (w_{00}+w_{01}U)[A^{\dagger}_{MN}f_z(V) - f_w(U)A^{\dagger}_{MN}](z_{00}+z_{01}V).$$

Noting (3) and $P_{(MN)} = 0$,we obtain

$$A^{\dagger}_{MN}f_z(V) - f_w(U)A^{\dagger}_{MN} = A^{\dagger}_{MN}f_z(V)P_{(MN)*} - A^{\dagger}_{MN}[Af_w(U) - f_z(V)A]A^{\dagger}_{MN}.$$

Because $rank[Af_w(U) - f_z(V)A] = rank[a^T(V,U)A]$,so

$$rank[a(U,V)A^{\dagger}_{MN}] \leq rank[A^{\dagger}_{MN}f_z(V)P_{(MN)*}] + rank[a^T(V,U)A].$$

Now we estimate the upper bound of $rank[A^{\dagger}_{MN}f_z(V)P_{(MN)*}]$.In fact,

$$rank[A^{\dagger}_{MN}f_z(V)P_{(MN)*}] = rank[Q_{(MN)*}f_z(V)P_{(MN)*}] = rank[Q_{(MN)*}VP_{(MN)*}] = rank(K).$$

The same as the case for A has full row rank and $m<n$,we obtain

$$rank(P_{(MN)}UA^{\dagger}_{MN}) \leq n-m.$$

Theorem 6.4.1. Let $A \in C^{m\times n}$ be of full rank,then

$$rank[a(U,V)A^{\dagger}_{MN}] \leq min\{m, n, rank[a^T(V,U)A] + |m-n|\}.$$

□

The upper bound can be attained.For example,let T be a 20×12 full column rank Toeplitz matrix,

$$a = (3, 2, 3, 4, 5, 1, 2, 3, 5, 3, 2, 7)$$

be the first row of T and

$$b = (3, 4, 2, 3, 4, 6, 2, 5, 3, 4, 5, 6, 1, 2, 3, 6, 7, 8, 3, 4)^T$$

be the first column of T.The weight matrices

$$M = diag(1, 2, 7, 5, 6, 2, 4, 3, 4, 6, 4, 8, 4, 2, 2, 5, 6, 2, 5, 4),$$

and

$$N = diag(4, 1, 1, 5, 3, 6, 4, 8, 7, 8, 4, 5).$$

We obtain

$$rank(T^{\dagger}_{MN} - Z_n T^{\dagger}_{MN} Z^*_m) = 10 = rank(T - Z^*_m T Z_n) + m - n.$$

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