Determinants

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1 Motivation

Determinant is a function that each square real matrix A is assigned a real number, denoted $\det A$, satisfying certain properties.

If A is a 3×3 matrix, writing $A = [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$, we require the absolute value of the determinant $\det A$ to be the volume of the parallelepiped spanned by the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

Since volume is linear in each side of the parallelepiped, it is required that the determinant function $\det A$ satisfies the following properties:

(a) **Linearity in Each Column**:

$$\det[\boldsymbol{u}_1 + \boldsymbol{u}_2, \boldsymbol{v}, \boldsymbol{w}] = \det[\boldsymbol{u}_1, \boldsymbol{v}, \boldsymbol{w}] + \det[\boldsymbol{u}_2, \boldsymbol{v}, \boldsymbol{w}],$$

$$\det[c\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = c \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}];$$

$$\det[\boldsymbol{u}, \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{w}] = \det[\boldsymbol{u}, \boldsymbol{v}_1, \boldsymbol{w}] + \det[\boldsymbol{u}, \boldsymbol{v}_2, \boldsymbol{w}],$$

$$\det[\boldsymbol{u}, c\boldsymbol{v}, \boldsymbol{w}] = c \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}];$$

$$\det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}_1 + \boldsymbol{w}_2] = \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}_1] + \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}_2],$$

$$\det[\boldsymbol{u}, \boldsymbol{v}, c\boldsymbol{w}] = c \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}].$$

(b) **Vanishing Property**: If two columns are the same, then the determinant is zero, i.e.,

$$\det[\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}] = \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}] = \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}] = 0.$$

(Actually, applying (a), it is easy to see that (b) is equivalent to requiring that $\det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = 0$ whenever $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent.)

(c) **Regularity**: Determinant of identity matrix is 1. (The volume of unit cube is 1.)

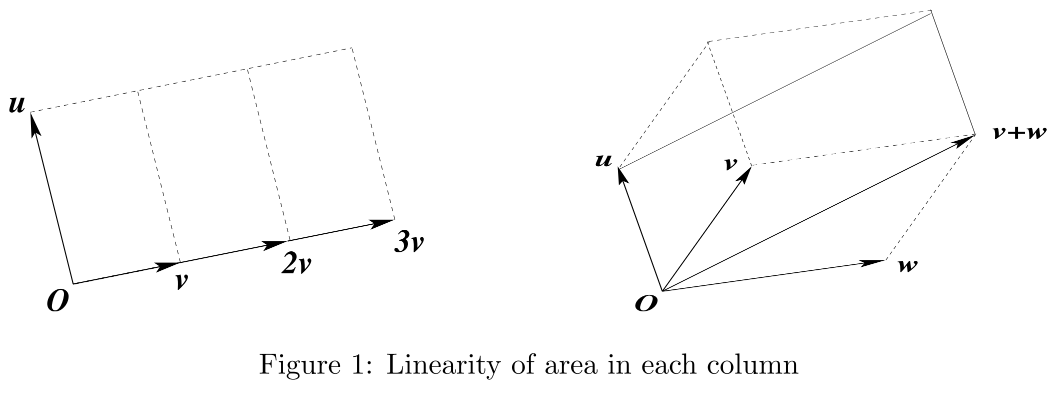

Figure 1: Linearity of area in each column

Definition 1.1. For an $n \times n$ matrix $A = [a_{ij}]$, the **determinant** of A is a real number $\det A$, satisfying the properties:

1. **Linearity**:

$$\det[\boldsymbol{v}_1, \ldots, a\boldsymbol{u}_i + b\boldsymbol{v}_i, \ldots, \boldsymbol{v}_n] = a \det[\boldsymbol{v}_1, \ldots, \boldsymbol{u}_i, \ldots, \boldsymbol{v}_n] + b \det[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_n];$$

2. **Vanishing Property**:

$$\det[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_n] = 0;$$

or equivalently,

$$\det[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_n] = -\det[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_n].$$

3. **Regularity**: $\det I_n = 1$.

2 Determinant of 1×1 and 2×2 matrices

For 1×1 matrices $[a_{11}]$ we must have

$$\det[a_{11}] = a_{11} \det[1] = a_{11} \cdot 1 = a_{11}.$$

The **determinant** for a 2×2 matrix $A = [a_{ij}]$ must be defined as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

In fact, write $A = [\boldsymbol{a}_1, \boldsymbol{a}_2]$. Then

$$\boldsymbol{a}_1 = a_{11}\boldsymbol{e}_1 + a_{21}\boldsymbol{e}_2, \quad \boldsymbol{a}_2 = a_{12}\boldsymbol{e}_1 + a_{22}\boldsymbol{e}_2.$$

Thus

$$\begin{aligned} \det A &= \det[a_{11}\boldsymbol{e}_1 + a_{21}\boldsymbol{e}_2, \boldsymbol{a}_2] \\ &= a_{11} \det[\boldsymbol{e}_1, \boldsymbol{a}_2] + a_{21} \det[\boldsymbol{e}_2, \boldsymbol{a}_2] \\ &= a_{11}a_{12} \det[\boldsymbol{e}_1, \boldsymbol{e}_1] + a_{11}a_{22} \det[\boldsymbol{e}_1, \boldsymbol{e}_2] + \\ &\quad a_{21}a_{12} \det[\boldsymbol{e}_2, \boldsymbol{e}_1] + a_{21}b_{22} \det[\boldsymbol{e}_2, \boldsymbol{e}_2]. \end{aligned}$$

Since $\det[\boldsymbol{e}_1, \boldsymbol{e}_2] = 1$, $\det[\boldsymbol{e}_1, \boldsymbol{e}_1] = \det[\boldsymbol{e}_2, \boldsymbol{e}_2] = 0$, and $\det[\boldsymbol{e}_2, \boldsymbol{e}_1] = -1$, we then have

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

3 Determinant of 3×3 matrices

The **determinant** for 3×3 matrices must be defined as

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{31}a_{22}a_{31}. \end{aligned}$$

4 Properties of determinant

Determinant can be equivalently defined by the same properties on rows instead of columns. In the following, we state such properties in detail.

1. Addition:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} + b_{i1} & \cdots & a_{in} + b_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ b_{i1} & \cdots & b_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

2. Scalar multiplication:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ ca_{i1} & \cdots & ca_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = c \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

3. If two rows are the same, then determinant is 0, i.e.,

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0.$$

4. Switching two rows changes sign of determinant, i.e.,

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

5. Adding a multiple of one row to another row does not change determinant, i.e.,

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1} + ca_{i1} & \cdots & a_{jn} + ca_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

Property 4 and Property 5 follow from Properties 1-3.

Proposition 4.1. *For triangular matrix,*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

Proof. If all $a_{11}, a_{22}, \ldots, a_{nn}$ are nonzero, then the matrix A can be reduced to a diagonal matrix whose diagonal entries are $a_{11}, a_{22}, \ldots, a_{nn}$. Thus

$$\det A = \det\left(\mathrm{Diag}[a_{11}, a_{22}, \ldots, a_{nn}]\right) = a_{11}a_{22}\cdots a_{nn}.$$

If some of $a_{11}, a_{22}, \ldots, a_{nn}$ are zero, we need to show $\det A = 0$. Let i be the smallest index such that $a_{ii} = 0$. If $i = 1$, then the first column of A is zero. Hence $\det A = 0$. So we may assume $i \geq 2$. Since $a_{11}, a_{22}, \ldots, a_{i-1,i-1}$ are nonzero, applying column operations the determinant of A equals the following:

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 & 0 & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{i-1,i-1} & 0 & \cdots & a_{i-1,n} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = 0.$$

□

Example 4.1.

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 4 & 3 \\ 1 & 2 & 1 \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=} - \begin{vmatrix} 1 & 2 & 1 \\ 5 & 4 & 3 \\ 2 & 3 & 4 \end{vmatrix} \overset{R_5 - 5R_1}{\underset{R_2 - 2R_1}{=}} - \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -2 \\ 0 & -1 & 2 \end{vmatrix} \overset{R_2 - 6R_3}{=} - \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & -14 \\ 0 & -1 & 2 \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{=} \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -14 \end{vmatrix} = 14.$$

Example 4.2. The **Vandermonde determinant** is

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} \overset{R_2 - a_1R_1}{\underset{R_3 - a_1R_2}{=}} & \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 \\ 0 & a_2^2 - a_1a_2 & a_3^2 - a_1a_3 \end{vmatrix} \\ = & \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 \\ 0 & a_2(a_2 - a_1) & a_3(a_3 - a_1) \end{vmatrix} \ R_3 - a_2R_2 \\ = & \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 \\ 0 & 0 & a_3(a_3 - a_1) - a_2(a_3 - a_1) \end{vmatrix} \\ = & \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 \\ 0 & 0 & (a_3 - a_1)(a_3 - a_2) \end{vmatrix} \\ = & (a_2 - a_1)(a_3 - a_1)(a_3 - a_2). \end{aligned}$$

5 Laplace expansion formula

For a 3×3 matrix $A = [a_{ij}]$, by the addition property,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Note that

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} + a_{11}a_{32} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 1 & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} - a_{11}a_{32} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & a_{33} \\ 0 & 0 & a_{23} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix};$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{12} & 0 & 0 \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix};$$

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{13} & 0 & 0 \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Thus

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Theorem 5.1 (Laplace Expansion). *Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then for any fixed i,*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}, \tag{5.1}$$

where A_{ij} is the $(n-1) \times (n-1)$ submatrix obtained from A by deleting the ith row and the jth column.

The identity (5.1) is known as the **Laplace expansion along the ith row**. The determinant $\det A_{ij}$ is called the **minor** of the entry a_{ij}. We define the **cofactor** of a_{ij} as

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

The **classical adjoint** of A is the matrix

$$\operatorname{adj} A := \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \left[C_{ij} \right]^T.$$

Example 5.1.

$$\begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 7 & 0 & 9 \\ -5 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 7 & 9 \\ -5 & -1 \end{vmatrix} = -7 + 45 = 38.$$

Theorem 5.2. *For any $n \times n$ matrix $A = [a_{ij}]$,*

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{cases} \det A & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In other words,

$$A \operatorname{adj} A = (\det A)\, I_n.$$

If $\det A \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Proof. Applying the cofactor expansion formula for A along the ith row, we have

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \det A.$$

Let M be the matrix obtained from A by replacing the jth row of A with the ith row of A. Then $\det M = 0$. Applying the cofactor expansion formula for M along the jth row, we have

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0, \quad i \neq j.$$

□

Corollary 5.3. *A square matrix A is invertible if and only if $\det A \neq 0$.*

Example 5.2. For the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix},$$

its adjoint matrix is given by

$$\operatorname{adj} A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Thus

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}.$$

6 Other properties of determinant

Theorem 6.1. *For any $n \times n$ matrices A and B,*

$$\det(AB) = \det A \det B.$$

Proof. The proof is divided into three steps.

STEP 1: $\det(AB) = \det A \det B$ if A is not invertible.

We claim that AB is not invertible. Suppose AB is invertible. Then there is a matrix C such that $(AB)C = I$, i.e., $A(BC) = I$. This means that A is invertible, a contradiction. Since $\det(AB) = \det B = 0$, we thus have $\det(AB) = \det A \det B$.

STEP 2: $\det(EB) = \det E \det B$ for elementary matrix E.

This follows from the interpretation for left multiplication of elementary matrices, and the properties of determinant.

STEP 3: $\det(AB) = \det A \det B$ if A is invertible.

Since A is invertible, it can be written as product of elementary matrices, i.e., $A = E_1 E_2 \cdots E_k$. Applying SEPT 2, we see that

$$\begin{aligned}\det(AB) &= \det(E_1 E_2 \cdots E_k B)\\ &= \det E_1 \det(E_2 \cdots E_k B)\\ &= \cdots\\ &= \det E_1 \det E_2 \cdots \det E_k \det B.\end{aligned}$$

Applying the same property,

$$\det A = \det E_1 \det E_2 \cdots \det E_k.$$

Hence $\det(AB) = \det A \det B$. □

Theorem 6.2. *For any square matrix A,*

$$\det A = \det A^T.$$

7 Cramer's rule

Theorem 7.1. *Let $A = [a_{ij}]$ be an invertible $n \times n$ matrix. Then the solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is given by*

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, 2, \ldots, n,$$

where A_i is matrix obtained from A by replacing the ith column with $\boldsymbol{b}$.

Proof. Let X_i be the matrix obtained from the identity matrix I_n by replacing the ith column with $\boldsymbol{x}$. Then

$$\begin{aligned}AX_i &= A\big[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{i-1}, \boldsymbol{x}, \boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_n\big]\\ &= \big[A\boldsymbol{e}_1, \ldots, A\boldsymbol{e}_{i-1}, A\boldsymbol{x}, A\boldsymbol{e}_{i+1}, \ldots, A\boldsymbol{e}_n\big]\\ &= \big[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{b}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_n\big] = A_i.\end{aligned}$$

Then $\det(AX_i) = \det A_i$. Note that

$$\det(AX_i) = \det A \det X_i = (\det A)x_i.$$

It follows that $x_i = \det A_i / \det A$. □

8 Algebraic formula of determinant

A **permutation** of $\{1, 2, \ldots, n\}$ is a bijection

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

The set of all such permutations are denoted by S_n; the number of such permutations is $n!$. A permutation σ can be described as

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix} \quad \text{or} \quad \sigma = j_1 j_2 \cdots j_n.$$

An **inversion** of σ is an ordered pair (j_i, j_k) of integers such that $j_i > j_k$. We denote by $\operatorname{inv}(\sigma)$ the set of all inversions of σ. A permutation σ is called **even** (**odd**) if the number of inversions of σ is an even (odd) number. We define

$$\operatorname{sgn} \sigma = (-1)^{\#\operatorname{inv}(\sigma)} = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

For instance, the permutation 3241 of $\{1, 2, 3, 4\}$ has inversions $(3, 2)$, $(3, 1)$, $(2, 1)$, $(4, 1)$; so 3241 is an even permutation. The permutation 2341 has inversions $(2, 1)$, $(3, 1)$, $(4, 1)$; so 2341 is an odd permutation.

Theorem 8.1. *For permutations $\sigma, \tau \in S_n$, we have*

$$\operatorname{sgn}(\tau \circ \sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma).$$

Moreover, $\operatorname{sgn}\sigma^{-1} = \operatorname{sgn}\sigma$.

Proof. Consider the polynomial

$$g = g(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j)$$

and the 1-dimensional vector space $\{cg \mid c \in \mathbb{R}\}$. For any cg, we define the polynomial

$$\sigma(cg) := c \prod_{i<j} \big(x_{\sigma(i)} - x_{\sigma(j)}\big).$$

Since σ is a permutation of $\{1, 2, \ldots, n\}$, we have $\sigma\big(\{1, 2, \ldots, n\}\big) = \{1, 2, \ldots, n\}$. Thus

$$\sigma(cg) = \pm c \prod_{i<j} \big(x_i - x_j\big) = (\operatorname{sgn}\sigma)cg.$$

Thus, on the one hand,

$$(\tau \circ \sigma)(g) = \big(\operatorname{sgn}(\tau \circ \sigma)\big)g.$$

On the other hand,

$$(\tau \circ \sigma)(g) = \tau\big(\sigma(g)\big) = \tau\big((\operatorname{sgn}\sigma)g\big) = (\operatorname{sgn}\tau)\operatorname{sgn}(\sigma)g.$$

It follows that $\operatorname{sgn}(\tau \circ \sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)$. □

Theorem 8.2. *For any $n \times n$ matrix $A = [a_{ij}]$,*

$$\begin{aligned} \det A &= \sum_{\sigma = i_1 i_2 \cdots i_n \in S_n} \operatorname{sgn}(\sigma) a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} \\ &= \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}. \end{aligned}$$

Proof. Let $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n]$. Then for each j,

$$\boldsymbol{a}_j = a_{1j}\boldsymbol{e}_1 + a_{2j}\boldsymbol{e}_2 + \cdots + a_{nj}\boldsymbol{e}_n = \sum_{i=1}^{n} a_{ij}\boldsymbol{e}_i.$$

By linearity, we have

$$\begin{aligned} \det A &= \det\Big[\sum a_{i_1,1}\boldsymbol{e}_{i_1}, \sum a_{i_2,1}\boldsymbol{e}_{i_2}, \ldots, \sum a_{i_n,n}\boldsymbol{e}_{i_n}\Big] \\ &= \sum_{i_1, i_2, \ldots, i_n} a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} \det[\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_n}]. \end{aligned}$$

If $i_1 i_2 \ldots i_n$ is not a permutation of $\{1, 2, \ldots, n\}$, then two of the vectors $\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_n}$ are the same, then

$$\det[\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_n}] = 0.$$

If $\sigma = i_1 i_2 \ldots i_n$ is a permutation of $\{1, 2, \ldots, n\}$, then

$$\det[\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_n}] = (-1)^{\#\mathrm{inv}(\sigma)} = \operatorname{sgn}\sigma.$$

Therefore,

$$\det A = \sum_{\sigma = i_1 i_2 \cdots i_n \in S_n} \operatorname{sgn}(\sigma) a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n}.$$

□

The above algebraic formula is sometimes used to define determinant.

Theorem 8.3. *For any $n \times n$ matrix $A = [a_{ij}]$,*

$$\det A^T = \det A.$$

Proof. Let $B = A^T = [b_{ij}]$. Then

$$\begin{aligned}
\det B &= \sum_{\sigma} (\operatorname{sgn}\sigma) b_{\sigma(1),1} b_{\sigma(2),2} \cdots b_{\sigma(n),n} \\
&= \sum_{\sigma} (\operatorname{sgn}\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \\
&= \sum_{\sigma} (\operatorname{sgn}\sigma) a_{\sigma^{-1}(1),1} a_{\sigma^{-1}(2),2} \cdots a_{\sigma^{-1}(n),n} \\
&= \sum_{\tau} (\operatorname{sgn}\tau^{-1}) a_{\tau(1),1} a_{\tau(2),2} \cdots a_{\tau(n),n} \\
&= \det A \quad (\text{as } \operatorname{sgn}\tau^{-1} = \operatorname{sgn}\tau).
\end{aligned}$$

□

Corollary 8.4. *For any $n \times n$ matrix $A = [a_{ij}]$,*

$$\begin{aligned}
\det A &= \sum_{\sigma = j_1 j_2 \cdots j_n \in S_n} \operatorname{sgn}(\sigma) a_{1j_1} a_{2j_2} \cdots a_{nj_n} \\
&= \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.
\end{aligned}$$

For a 3×3 matrix $A = [a_{ij}]$, since

$$\operatorname{sgn}(123) = \operatorname{sgn}(231) = \operatorname{sgn}(312) = 1,$$

$$\operatorname{sgn}(132) = \operatorname{sgn}(213) = \operatorname{sgn}(321) = -1,$$

we then have

$$\begin{aligned}
\det A &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
&\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.
\end{aligned}$$

Theorem 8.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the standard matrix A. Let P be a parallelotope spanned by the vector $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Then*

$$\operatorname{vol}\big(T(P)\big) = |\det(A)| \cdot \operatorname{vol}(P).$$

Proof. Let $\boldsymbol{v}_i = [v_{1i}, v_{2i}, \ldots, v_{ni}]^T$, i.e., $\boldsymbol{v}_i = v_{1i}\boldsymbol{e}_1 + v_{2i}\boldsymbol{e}_2 + \cdots + v_{ni}\boldsymbol{e}_n$ $(1 \le i \le n)$. Then

$$T(\boldsymbol{v}_i) = v_{1i}T(\boldsymbol{e}_1) + v_{2i}T(\boldsymbol{e}_2) + \cdots + v_{ni}T(\boldsymbol{e}_n).$$

Thus

$$\big[T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), \ldots, T(\boldsymbol{v}_n)\big] = \big[T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n)\big] \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \cdots & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} = A\big[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\big].$$

Therefore

$$\begin{aligned}
\operatorname{vol}\big(T(P)\big) &= \big|\det\big[T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), \ldots, T(\boldsymbol{v}_n)\big]\big| \\
&= \big|\det(A)\big| \cdot \big|\det\big[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\big]\big| \\
&= \big|\det(A)\big| \cdot \big|\operatorname{vol}(P).
\end{aligned}$$

□