Homework 3

§4.1: 2, 8, 16, 18, 20 §4.2: 6, 8, 24, 32 §4.3: 10, 14, 20, 24
§4.4: 4, 8, 14, 30 §4.5: 12, 14, 30 §4.6: 4, 6, 14, 28

Supplementary exercises for determinants

1. Let $\sigma = 364152, \tau = 246513, \rho = 413562$ be permutations of $\{1,2,3,4,5,6\}$.
   (a) Find parity of σ, τ, ρ.
   (b) Find $\tau \circ \sigma$, $\rho \circ \tau \circ \sigma$, and σ^{-1}.
2. Let $g = g(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_i)$. Let
$$\sigma(g) = \prod_{i<j} \big(x_{\sigma(i)} - x_{\sigma(j)}\big).$$
   Show that $\sigma(g) = (\operatorname{sgn} \sigma) g$.
3. Find the determinant of each of the following matrices.
$$A = \begin{bmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 & 4 \\ 6 & -3 & -2 \\ 4 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{bmatrix}$$
   Then find the $\operatorname{adj} A$, $\operatorname{adj} B$, A^{-1}, and B^{-1} if A and B are invertible.
4. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 - x_2 + x_3, -x_1 + x_2 + x_3)$. Let P be the parallelepiped spanned by the three vectors
$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$
   i.e., $P = \big\{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 \,|\, 0 \le c_1, c_2, c_3 \le 1\big\}$. Find the volume of P and the volume of
$$T(P) = \big\{T(\boldsymbol{x}) \,|\, \boldsymbol{x} \in P\big\}.$$

Supplementary exercises for matrices of linear transformations

1. Let $\mathbb{P}_n(t)$ be the vector space of all polynomials of degree at most n. Let
$$\mathcal{B} = \big\{1, t, t(t+1), t(t+1)(t+2)\big\}, \quad \mathcal{B}' = \big\{1, t, t(t-1), t(t-1)(t-2)\big\}.$$
   (a) Show that $\mathcal{B}$ and $\mathcal{B}'$ are bases of $\mathbb{P}_3(t)$.
   (b) Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.
   (c) Show that $T : \mathbb{P}_3(t) \longrightarrow \mathbb{P}_3(t)$, defined by $T\big(p(t)\big) = p(t) + tp'(t)$, is a linear transformation.
   (d) Find the matrix A of T relative to the basis $\mathcal{B}$, and the matrix B of T relative to the basis $\mathcal{B}'$.
   (e) Find the relation between the matrices A and B.
2. Let $\mathbb{M}_{m,n}$ be the vector space of all $m \times n$ matrices. Note that
$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
   is a basis of $\mathbb{M}_{3,2}$, and
$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
   is a basis of $\mathbb{M}_{2,2}$.

(a) Show that the set

$$\mathcal{B}' = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of $\mathbb{M}_{3,2}$, and that

$$\mathcal{C}' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

is a basis of $\mathbb{M}_{2,2}$.

(b) Show that $F : \mathbb{M}_{3,2} \longrightarrow \mathbb{M}_{2,2}$, defined by

$$F\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix},$$

is a linear transformation.

(c) Find the matrix A of F relative to the bases $\mathcal{B}$ and $\mathcal{C}$, and the matrix B of F relative to the bases $\mathcal{B}'$ and $\mathcal{C}'$.

(d) Find the relation between the matrices A and B.

3. Let $F : \mathbb{M}_{2,2} \longrightarrow \mathbb{P}_2(t)$ be defined by

$$F\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \right) = (x_{11} + x_{12}) + (x_{12} + x_{21})t + (x_{21} + x_{22})t^2.$$

(a) Find the matrix A of F relative to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of $\mathbb{M}_{2,2}$ and the basis $\mathcal{C} = \{1, t, t^2\}$ of $\mathbb{P}_2$.

(b) Find the matrix B of F relative to the basis

$$\mathcal{B}' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

of $\mathbb{M}_{2,2}$ and the basis $\mathcal{C}' = \{1, t, t(t+1)\}$ of $\mathbb{P}_2$.

(c) Find the relation between A and B.

4. Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 + x_4, x_2 - x_3 + x_5, 3x_1 + 3x_3 - 2x_4 + 2x_5).$$

(a) Find the matrix B of T relative to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^5$ and the basis

$$\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$.

(b) Let V be the subspace of $\mathbb{R}^5$ defined by the linear system

$$(*)\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_1 + x_2 - x_3 - x_4 + x_5 = 0 \end{cases},$$

and let W be the subspace of $\mathbb{R}^3$ defined by the linear equation $2x_1 + 3x_2 + x_3 = 0$. Show that T defines a linear transformation from V to W.

(c) Find the matrix of T from V to W relative to the basis $\mathcal{B}$ of V, consisting of the basic solutions of the linear system (*), and the basis $\mathcal{C}$ of W, consisting of the basic solutions of the linear equation $2x_1 + 3x_2 + x_3 = 0$.

5. Let V be an n-dimensional vector space, and let W be an m-dimensional vector space. Let $\mathrm{Hom}(V, W)$ denote the set of all linear transformations from V to W. For $F, G \in \mathrm{Hom}(V, W)$, define the **addition** and **scalar multiplication** as

$$(F + G)(\boldsymbol{v}) = F(\boldsymbol{v}) + G(\boldsymbol{v}),$$
$$(cF)(\boldsymbol{v}) = cF(\boldsymbol{v}).$$

(a) Show that $\mathrm{Hom}(V, W)$ is a vector space.

(b) Given a basis $\mathcal{B}$ of V and a basis $\mathcal{C}$ of W. Let $T : \mathrm{Hom}(V, W) \longrightarrow \mathbb{M}_{m,n}$ be defined by

$$T(F) = \text{the matrix of } F \text{ relative to the bases } \mathcal{B} \text{ and } \mathcal{C}.$$

Show that T is a one-to-one and onto linear transformation.

6. Let V be the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form $f(t) = (a_0 + a_1 t + a_2 t^2)e^{2t}$, where $a_0, a_1, a_3 \in \mathbb{R}$.

(a) Show that V is a subspace of the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(b) Let $D : V \longrightarrow V$ be defined by $D\big(f(t)\big) = f'(t)$. Find the matrix of D relative to the basis $\{e^{2t}, te^{2t}, t^2e^{2t}\}$.

(c) Is D invertible? If yes, find the inverse transformation of D.