Homework 4

§5.1: 6, 8, 16; §5.2: 12, 18; §5.3: 6, 16, 24; §6.1: 16, 24; §6.2: 6, 10, 20; §6.3: 2, 10, 14; §6.4: 12

1. Consider the matrix
$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$
   (a) Diagonalize A;
   (b) Use the diagonalization to find A^{10};
   (c) Determine the invertibility of A from its diagonalization;
   (d) Use the diagonalization to compute $A^5 - 2A^4 + A^3$.

2. Determine whether the following statements are true or false (no reason needed).
   (a) If all eigenvalues of A are 0, then $A = 0$.
   (b) If $A^9 = 0$, then all eigenvalues of A are 0.
   (c) If $\boldsymbol{v}$ is an eigenvector of A, then $\boldsymbol{v}$ is an eigenvector of A^2.
   (d) If $\boldsymbol{v}$ is an eigenvector of A, then $\boldsymbol{v}$ is an eigenvector of A^T.
   (e) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are eigenvectors of A, then $\mathbf{u} + \mathbf{v}$ is also an eigenvector of A.
   (f) If $\boldsymbol{u}$ is an eigenvector of A, so is $2\boldsymbol{u}$.
   (g) If A and B are diagonalizable, so is AB.
   (h) If A is diagonalizable and invertible, then A^{-1} is also diagonalizable.
   (i) Similar matrices have the same eigenvalues.
   (j) If two matrices have the same eigenvalues, then they are similar.

3. Consider the matrix
$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 1 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
   (a) Find an orthogonal basis of $\operatorname{Col} A$;
   (b) Find the orthogonal projection of $\boldsymbol{u}$ on $\operatorname{Col} A$;
   (c) Find the distance from $\boldsymbol{u}$ to $\operatorname{Col} A$.

4. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which magnifies 3 times in the direction $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 5 times in the direction $\boldsymbol{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

5. Find numbers a, b, c such that
$$U = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$$
is an orthogonal matrix. Then find U^{-1}.

6. Determine whether the following statements are true or false (no reason needed).
   (a) Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$ be vectors of $\mathbb{R}^n$. If $\boldsymbol{v}_1$ is orthogonal to $\boldsymbol{v}_3$ and $\boldsymbol{v}_2$ is orthogonal to $\boldsymbol{v}_4$, then $\boldsymbol{v}_1 + \boldsymbol{v}_2$ is orthogonal to $\boldsymbol{v}_3 + \boldsymbol{v}_4$.
   (b) If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthonormal set, then it is linearly independent.

(c) If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthogonal basis of $\mathbb{R}^3$, then $\{3\boldsymbol{v}_1, -2\boldsymbol{v}_2, 5\boldsymbol{v}_3\}$ is also an orthogonal basis.

(d) If $\boldsymbol{u} \perp \boldsymbol{v}$, $\boldsymbol{v} \perp \boldsymbol{w}$, then $\boldsymbol{u} \perp \boldsymbol{w}$.

(e) If $\boldsymbol{u} \in W$ and $\boldsymbol{u} \in W^{\perp}$, then $\boldsymbol{u} = 0$.

(f) If the orthogonal projection of $\boldsymbol{y}$ onto W is $\hat{\boldsymbol{y}}$, then the orthogonal projection of $8\boldsymbol{y}$ is $8\hat{\boldsymbol{y}}$.

(g) If the column vectors of a matrix is orthogonal, so are the row vectors.

(h) If the row vectors of a square matrix is orthonormal, so are the column vectors.

(i) If U and V are orthogonal matrices, so is $U + V$.

(j) If U and V are orthogonal matrices, so is UV^{-1}.

7. Let $\mathbb{P}_2$ be the vector space of all real polynomials of degree at most 2. For $p(t), q(t) \in \mathbb{P}_2$, let

$$\langle p(t), q(t) \rangle = p(0)q(0) + p(0)q'(0) + p'(0)q(0) + 2p'(0)q'(0) + p''(0)q''(0).$$

(a) Show that $\langle , \rangle$ is an inner product on the vector space $\mathbb{P}_2$.

(b) Let $f(t) = 1 + 2t + 3t^2$, and let $\mathrm{Proj}_{f(t)} : \mathbb{P}_2 \to \mathbb{P}_2$ be the orthogonal projection along the vector $f(t)$. Find the matrix of $\mathrm{Proj}_{f(t)}$ relative to the basis $\{1, t, t^2\}$ of $\mathbb{P}_2$.

(c) Find all eigenvalues of $\mathrm{Proj}_{f(t)}$.

(d) For each eigenvalues of $\mathrm{Proj}_{f(t)}$, find its independent eigenvectors.

8. Let W be a subspace of $\mathbb{R}^3$ defined by the equation $x_1 - x_2 + x_3 = 0$. Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the range of T is W and

$$\|T(\boldsymbol{x})\| = \|\boldsymbol{x}\| \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^3.$$

9. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

10. Let W be the subspace of $\mathbb{R}^4$ defined by the linear system $\left\{ \begin{array}{l} x_1 - x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \end{array} \right.$.

(a) Find the standard matrix of the orthogonal projection $\mathrm{Proj}_W : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ of the Euclidean space $\mathbb{R}^4$ with the dot product.

(b) Find all eigenvalues of Proj_W.

(c) For each eigenvalues of Proj_W, find its independent eigenvectors.