Systems of Linear Equations

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1 Systems of linear equations

Linear systems

A **linear equation** in variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are constant real or complex numbers. The constant a_i is called the **coefficient** of x_i; and b is called the **constant term** of the equation.

A **system of linear equations** (or **linear system**) is a finite collection of linear equations in same variables. For instance, a linear system of m equations in n variables $x_1, x_2, \ldots, x_n$ can be written as

$$\left\{\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}\right. \tag{1.1}$$

A **solution** of a linear system (1.1) is a tuple $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively. The set of all solutions of a linear system is called the **solution set** of the system.

Theorem 1.1. *Any system of linear equations has one of the following exclusive conclusions.*

(a) *No solution.*

(b) *Unique solution.*

(c) *Infinitely many solutions.*

A linear system is said to be **consistent** if it has at least one solution; and is said to be **inconsistent** if it has no solution.

Geometric interpretation

The following three linear systems

$$(a)\left\{\begin{array}{rrcr} 2x_1 & +x_2 & = & 3 \\ 2x_1 & -x_2 & = & 0 \\ x_1 & -2x_2 & = & 4 \end{array}\right. \quad (b)\left\{\begin{array}{rrcr} 2x_1 & +x_2 & = & 3 \\ 2x_1 & -x_2 & = & 5 \\ x_1 & -2x_2 & = & 4 \end{array}\right. \quad (c)\left\{\begin{array}{rrcr} 2x_1 & +x_2 & = & 3 \\ 4x_1 & +2x_2 & = & 6 \\ 6x_1 & +3x_2 & = & 9 \end{array}\right.$$

have no solution, a unique solution, and infinitely many solutions, respectively. See Figure 1.

Note: A linear equation of two variables represents a straight line in $\mathbb{R}^2$. A linear equation of three variables represents a plane in $\mathbb{R}^3$.In general, a linear equation of n variables represents a hyperplane in the n-dimensional Euclidean space $\mathbb{R}^n$.

Matrices of a linear system

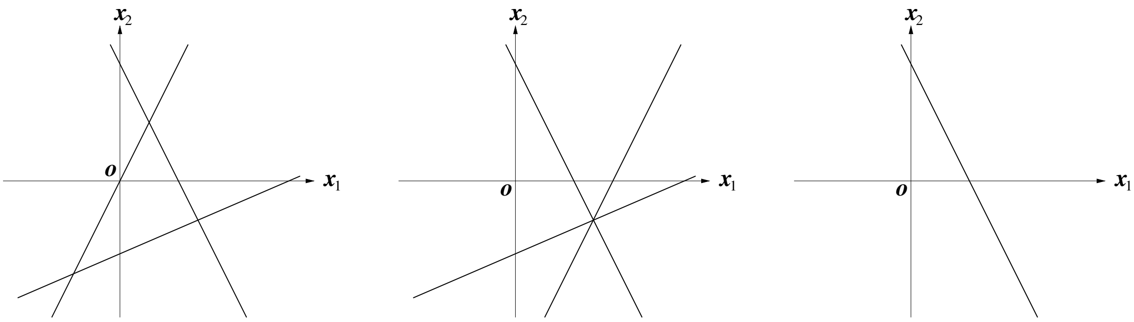

Figure 1: No solution, unique solution, and infinitely many solutions.

Definition 1.2. The **augmented matrix** of the general linear system (1.1) is the table

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] \tag{1.2}$$

and the **coefficient matrix** of (1.1) is

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{1.3}$$

Systems of linear equations can be represented by matrices. Operations on equations (for eliminating variables) can be represented by appropriate row operations on the corresponding matrices. For example,

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ 2x_1 & -3x_2 & +x_3 & = & -8 \\ 3x_1 & +x_2 & +4x_3 & = & 7 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 2 & -3 & 1 & -8 \\ 3 & 1 & 4 & 7 \end{array}\right]$$

$$\begin{array}{c} [\mathrm{Eq}\,2] - 2[\mathrm{Eq}\,1] \\ [\mathrm{Eq}\,3] - 3[\mathrm{Eq}\,1] \end{array} \qquad \begin{array}{c} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & -5x_2 & +5x_3 & = & -10 \\ & -2x_2 & +10x_3 & = & 4 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & -5 & 5 & -10 \\ 0 & -2 & 10 & 4 \end{array}\right]$$

$$\begin{array}{c} (-1/5)[\mathrm{Eq}\,2] \\ (-1/2)[\mathrm{Eq}\,3] \end{array} \qquad \begin{array}{c} (-1/5)R_2 \\ (-1/2)R_3 \end{array}$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & x_2 & -x_3 & = & 2 \\ & x_2 & -5x_3 & = & -2 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -5 & -2 \end{array}\right]$$

$$[\mathrm{Eq}\,3] - [\mathrm{Eq}\,2] \qquad R_3 - R_2$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & x_2 & -x_3 & = & 2 \\ & & -4x_3 & = & -4 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

$$(-1/4)[\mathrm{Eq}\,3] \qquad (-1/4)R_3$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & -2x_3 & = & 1 \\ & x_2 & -x_3 & = & 2 \\ & & x_3 & = & 1 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} [\mathrm{Eq}\,1] + 2[\mathrm{Eq}\,3] \\ [\mathrm{Eq}\,2] + [\mathrm{Eq}\,3] \end{array} \qquad \begin{array}{c} R_1 + 2R_3 \\ R_2 + R_3 \end{array}$$

$$\left\{\begin{array}{rrrcr} x_1 & +x_2 & & = & 3 \\ & x_2 & & = & 3 \\ & & x_3 & = & 1 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$[\mathrm{Eq}\,1] - [\mathrm{Eq}\,2] \qquad R_1 - R_2$$

$$\left\{\begin{array}{rrrcr} x_1 & & & = & 0 \\ & x_2 & & = & 3 \\ & & x_3 & = & 1 \end{array}\right. \qquad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Elementary row operations

Definition 1.3. There are three kinds of elementary row operations on matrices:

(a) Adding a multiple of one row to another row;

(b) Multiplying all entries of one row by a nonzero constant;

(c) Interchanging two rows.

Definition 1.4. Two linear systems in same variables are said to be **equivalent** if their solution sets are the same. A matrix A is said to be **row equivalent** to a matrix B , written

$$A \sim B,$$

if there is a sequence of elementary row operations that changes A to B.

Theorem 1.5. *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set. In other words, elementary row operations do not change solution set.*

Proof. It is trivial for the row operations (b) and (c) in Definition 1.3. Consider the row operation (a) in Definition 1.3. Without loss of generality, we may assume that a multiple of the first row is added to the second row. Let us only exhibit the first two rows as follows

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] \tag{1.4}$$

Do the row operation $(\text{Row}\,2) + c(\text{Row}\,1)$. We obtain

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21}+ca_{11} & a_{22}+ca_{22} & \dots & a_{2n}+ca_{2n} & b_2+cb_1 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] \tag{1.5}$$

Let $(s_1, s_2, \dots, s_n)$ be a solution of (1.4), that is,

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i, \quad 1 \le i \le m. \tag{1.6}$$

In particular,

$$a_{11}s_1 + a_{12}x_2 + \cdots + a_{1n}s_n \;=\; b_1, \tag{1.7}$$

$$a_{21}s_1 + a_{21}x_2 + \cdots + a_{2n}s_n \;=\; b_2. \tag{1.8}$$

Multiplying c to both sides of (1.7), we have

$$ca_{11}s_1 + ca_{12} + \cdots + ca_{1n}s_n = cb_1. \tag{1.9}$$

Adding both sides of (1.8) and (1.9), we obtain

$$(a_{21}+ca_{11})s_1 + (a_{22}+ca_{12})s_2 + \cdots + (a_{2n}+ca_{1n})s_n = b_2 + cb_1. \tag{1.10}$$

This means that $(s_1, s_2, \dots, s_n)$ is a solution of (1.5).

Conversely, let $(s_1, s_2, \dots, s_n)$ be a solution of (1.5), i.e., (1.10) is satisfied and the equations of (1.6) are satisfied except for $i = 2$. Since

$$a_{11}s_1 + a_{12}x_2 + \cdots + a_{1n}s_n = b_1,$$

multiplying c to both sides we have

$$c(a_{11}s_1 + a_{12} + \cdots + a_{1n}s_n) = cb_1. \tag{1.11}$$

Note that (1.10) can be written as

$$(a_{21}s_1 + a_{22}s_2 + \cdots + a_{2n}s_n) + c(a_{11}s_1 + a_{12}s_2 + \cdots + a_{1n}s_n) = b_2 + cb_1. \tag{1.12}$$

Subtracting (1.11) from (1.12), we have

$$a_{21}s_1 + a_{22}s_2 + \cdots + a_{2n}s_n = b_2.$$

This means that $(s_1, s_2, \dots, s_n)$ is a solution of (1.4). □

2 Row echelon forms

Definition 2.1. A matrix is said to be in **row echelon form** if it satisfies the following two conditions:

(a) All zero rows are gathered near the bottom.

(b) The first nonzero entry of a row, called the **leading entry** of that row, is ahead of the first nonzero entry of the next row.

A matrix in row echelon form is said to be in **reduced row echelon form** if it satisfies two more conditions:

(c) The leading entry of every nonzero row is 1.

(d) Each leading entry 1 is the only nonzero entry in its column.

A matrix in (reduced) row echelon form is called a **(reduced) row echelon matrix**.

Note 1. Sometimes we call row echelon forms just as echelon forms and row echelon matrices as echelon matrices without mentioning the word "row."

Row echelon form pattern

The following are two typical row echelon matrices.

$$\begin{bmatrix} \bullet & * & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the circled stars $\bullet$ represent arbitrary nonzero numbers, and the stars $*$ represent arbitrary numbers, including zero. The following are two typical reduced row echelon matrices.

$$\begin{bmatrix} 1 & 0 & * & * & 0 & * & 0 & * & * \\ 0 & 1 & * & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 2.2. If a matrix A is row equivalent to a row echelon matrix B, we say that A has the **row echelon form** B; if B is further a reduced row echelon matrix, then we say that A has the **reduced row echelon form** B.

Row reduction algorithm

Definition 2.3. A **pivot position** of a matrix A is a location of entries of A that corresponds to a leading entry in a row echelon form of A. A **pivot column** (**pivot row**) is a column (row) of A that contains a pivot position.

Algorithm 2.1 (Row Reduction Algorithm). (1) Begin with the leftmost nonzero column, which is a **pivot column**; the top entry is a **pivot position**.

(2) If the entry of the pivot position is zero, select a nonzero entry in the pivot column, interchange the pivot row and the row containing this nonzero entry.

(3) If the pivot position is nonzero, use elementary row operations to reduce all entries below the pivot position to zero, (and the pivot position to 1 and entries above the pivot position to zero for reduced row echelon form).

(4) Cover the **pivot row** and the rows above it; repeat (1)-(3) to the remaining submatrix.

Theorem 2.4. *Every matrix is row equivalent to one and only one reduced row echelon matrix. In other words, every matrix has a unique reduced row echelon form.*

Proof. The Row Reduction Algorithm show the existence of reduced row echelon matrix for any matrix M. We only need to show the uniqueness. Suppose A and B are two reduced row echelon forms for a matrix M. Then the systems $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$ have the same solution set. Write $A = [a_{ij}]$ and $B = [b_{ij}]$.

We first show that A and B have the same pivot columns. Let $i_1, \ldots, i_k$ be the pivot columns of A, and let $j_1, \ldots, j_l$ be the pivot columns of B. Suppose $i_1 = j_1, \ldots, i_{r-1} = j_{r-1}$, but $i_r \neq j_r$. Assume $i_r < j_r$. Then the i_rth row of A is

$$\big[0,\ \ldots,\ 0,\ 1,\ a_{r,i_r+1},\ \ldots, a_{r,j_r},\ a_{r,j_r+1},\ \ldots,\ a_{r,n}\big].$$

While the j_rth row of B is

$$\big[0,\ \ldots,\ 0,\ 1,\ b_{r,j_r+1},\ \ldots,\ b_{r,n}\big].$$

Since $i_{r-1} = j_{r-1}$ and $i_r < j_r$, we have $j_{r-1} < i_r < j_r$. So x_{i_r} is a free variable for $B\boldsymbol{x} = \mathbf{0}$. Let

$$u_{i_1} = -b_{1,i_r}, \quad \ldots, \quad u_{i_{r-1}} = -b_{r-1,i_r}, \quad u_{i_r} = 1, \quad \text{and} \quad u_i = 0 \quad \text{for} \quad i > i_r.$$

Then $\boldsymbol{u}$ is a solution of $B\boldsymbol{x} = \mathbf{0}$, but is not a solution of $A\boldsymbol{x} = \mathbf{0}$. This is a contradiction. Of course, $k = l$.

Next we show that for $1 \leq r \leq k = l$, we have

$$a_{rj} = b_{rj}, \quad j_r + 1 \leq j \leq j_{r+1} - 1.$$

Otherwise, we have $a_{r_0 j_0} \neq b_{r_0 j_0}$ such that r_0 is smallest and then j_0 is smallest. Set

$$u_{j_0} = 1, \quad u_{i_1} = -a_{1,j_0}, \quad \ldots, \quad u_{r_0} = -a_{r_0 j_0}, \quad \text{and} \quad u_j = 0 \quad \text{otherwise}.$$

Then $\boldsymbol{u}$ is a solution of $A\boldsymbol{x} = \mathbf{0}$, but is not a solution of $B\boldsymbol{x} = \mathbf{0}$. This is a contradiction. □

Solving linear system

Example 2.1. Find all solutions for the linear system

$$\begin{cases} x_1 & +2x_2 & -x_3 & = & 1 \\ 2x_1 & +x_2 & +4x_3 & = & 2 \\ 3x_1 & +3x_2 & +4x_3 & = & 1 \end{cases}$$

Solution. Perform the row operations:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ \sim \\ R_3 - 3R_1 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 7 & -2 \end{array}\right] \begin{array}{c} (-1/3)R_2 \\ \sim \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ \sim \\ R_2 + 2R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system is equivalent to

$$\begin{cases} x_1 & = & 7 \\ x_2 & = & -4 \\ x_3 & = & -2 \end{cases}$$

which means the system has a unique solution.

Example 2.2. Solve the linear system

$$\left\{\begin{array}{rrrrcr} x_1 & -x_2 & +x_3 & -x_4 & = & 2 \\ x_1 & -x_2 & +x_3 & +x_4 & = & 0 \\ 4x_1 & -4x_2 & +4x_3 & & = & 4 \\ -2x_1 & +2x_2 & -2x_3 & +x_4 & = & -3 \end{array}\right.$$

Solution. Do the row operations:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{array}\right] \begin{array}{c} R_2 - R_1 \\ R_3 - 4R_1 \\ \sim \\ R_4 + 2R_1 \end{array} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} (1/2)R_2 \\ R_3 - 2R_2 \\ \sim \\ R_4 + (1/2)R_2 \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \sim \end{array} \left[\begin{array}{cccc|r} (1) & [-1] & [1] & 0 & 1 \\ 0 & 0 & 0 & (1) & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The linear system is equivalent to

$$\left\{\begin{array}{lcl} x_1 & = & 1 + x_2 - x_3 \\ x_4 & = & -1 \end{array}\right.$$

We see that the variables x_2, x_3 can take arbitrary numbers; they are called **free variables**. Let $x_2 = c_1$, $x_3 = c_2$, where $c_1, c_2 \in \mathbb{R}$. Then $x_1 = 1 + c_1 - c_2$, $x_4 = -1$. All solutions of the system are given by

$$\left\{\begin{array}{lcl} x_1 & = & 1 + c_1 - c_2 \\ x_2 & = & c_1 \\ x_3 & = & c_2 \\ x_4 & = & -1 \end{array}\right.$$

The general solutions may be written as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where } c_1, c_2 \in \mathbb{R}.$$

Set $c_1 = c_2 = 0$, i.e., set $x_2 = x_3 = 0$, we have a **particular solution**

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

For the **corresponding homogeneous linear system** $A\boldsymbol{x} = \boldsymbol{0}$, i.e.,

$$\left\{\begin{array}{rrrrcr} x_1 & -x_2 & +x_3 & -x_4 & = & 0 \\ x_1 & -x_2 & +x_3 & +x_4 & = & 0 \\ 4x_1 & -4x_2 & +4x_3 & & = & 0 \\ -2x_1 & +2x_2 & -2x_3 & +x_4 & = & 0 \end{array}\right.$$

we have

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 0 \\ -2 & 2 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccr|r} (1) & [-1] & [1] & -1 & 0 \\ 0 & 0 & 0 & (1) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Set $c_1 = 1, c_2 = 0$, i.e., $x_2 = 1, x_3 = 0$, we obtain one **basic solution**

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

for the homogeneous system $A\boldsymbol{x} = \mathbf{0}$.

Set $c_1 = 0, c_2 = 1$, i.e., $x_2 = 0, x_3 = 1$, we obtain another **basic solution**

$$\boldsymbol{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

for the homogeneous system $A\boldsymbol{x} = \mathbf{0}$.

Example 2.3. The linear system with the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 2 & 1 & 5 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right]$$

has no solution because its augmented matrix has the row echelon form

$$\left[\begin{array}{ccr|r} (1) & 2 & -1 & 1 \\ 0 & (-3) & [7] & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The last row represents a contradictory equation $0 = -2$.

Theorem 2.5. *A linear system is consistent if and only if the row echelon form of its augmented matrix contains no row of the form*

$$\big[0, \ldots, 0 \,\big|\, b\big], \quad \textit{where} \quad b \neq 0.$$

Example 2.4. Solve the linear system whose augmented matrix is

$$A = \left[\begin{array}{rrrrrr|r} 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 3 & 6 & 0 & 3 & -3 & 2 & 7 \\ 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 2 & 4 & -2 & 4 & -6 & -5 & -4 \end{array}\right]$$

Solution. Interchanging Row 1 and Row 2, we have

$$\left[\begin{array}{rrrrrr|r} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 3 & 6 & 0 & 3 & -3 & 2 & 7 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 2 & 4 & -2 & 4 & -6 & -5 & -4 \end{array}\right] \begin{array}{c} R_2 - 3R_1 \\ \sim \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{rrrrrr|r} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 2 & -4 & -5 & -6 \end{array}\right] \begin{array}{c} R_2 \leftrightarrow R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{rrrrrr|r} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -2 & 2 & -4 & -5 & -6 \end{array}\right] \begin{array}{c} R_4 + 2R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & -3 & -6 \end{array}\right] \begin{array}{c} R_4 + \frac{3}{2}R_3 \\ \sim \\ \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_2 - R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{cccccc|c} (1) & [2] & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & (1) & [-1] & [2] & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & (1) & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then the system is equivalent to

$$\begin{cases} x_1 = 1 - 2x_2 - x_4 + x_5 \\ x_3 = -2 + x_4 - 2x_5 \\ x_6 = 2 \end{cases}$$

The unknowns x_2, x_4 and x_5 are free variables.

Set $x_2 = c_1$, $x_4 = c_2$, $x_5 = c_3$, where c_1, c_2, c_3 are arbitrary. The general solutions of the system are given by

$$\begin{cases} x_1 = 1 - 2c_1 - c_2 + c_3 \\ x_2 = c_1 \\ x_3 = -2 + c_2 - 2c_3 \\ x_4 = c_2 \\ x_5 = c_3 \\ x_6 = 2 \end{cases}$$

The general solution may be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Definition 2.6. A variable in a consistent linear system is called **free** if its corresponding column in the coefficient matrix is not a pivot column.

Theorem 2.7. *For any homogeneous system $A\boldsymbol{x} = \mathbf{0}$,*

$$\#\{\text{variables}\} = \#\{\text{pivot positions of } A\} + \#\{\text{free variables}\}.$$

3 Vector Space $\mathbb{R}^n$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let $\mathbb{R}^2$ be the set of all ordered pairs (u_1, u_2) of real numbers, called the **2-dimensional Euclidean space**. Each ordered pair (u_1, u_2) is called a **point** in $\mathbb{R}^2$. For each point (u_1, u_2) we associate a column

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

called the **vector** associated to the point (u_1, u_2). The set of all such vectors is still denoted by $\mathbb{R}^2$. So by a point in the Euclidean space $\mathbb{R}^2$ we mean an ordered pair

$$(x, y)$$

and, by a vector in the vector space $\mathbb{R}^2$ we mean a column

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Similarly, we denote by $\mathbb{R}^3$ the set of all tuples

$$(u_1, u_2, u_3)$$

of real numbers, called **points** in the Euclidean space $\mathbb{R}^3$. We still denote by $\mathbb{R}^3$ the set of all columns

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

called **vectors** in $\mathbb{R}^3$. For example, $(2, 3, 1)$, $(-3, 1, 2)$, $(0, 0, 0)$ are points in $\mathbb{R}^3$, while

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are vectors in the vector space $\mathbb{R}^3$.

Definition 3.1. The addition, subtraction, and scalar multiplication for vectors in $\mathbb{R}^2$ are defined by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix},$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix},$$

$$c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Similarly, the addition, subtraction, and scalar multiplication for vectors in $\mathbb{R}^3$ are defined by

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix},$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix},$$

$$c \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}.$$

Vectors in $\mathbb{R}^n$

Definition 3.2. Let $\mathbb{R}^n$ be the set of all tuples $(u_1, u_2, \ldots, u_n)$ of real numbers, called **points** in the n-**dimensional Euclidean space** $\mathbb{R}^n$. We still use $\mathbb{R}^n$ to denote the set of all columns

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

of real numbers, called **vectors** in the n**-dimensional vector space** $\mathbb{R}^n$. The vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is called the **zero vector** in $\mathbb{R}^n$. The **addition** and the **scalar multiplication** in $\mathbb{R}^n$ are defined by

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_1 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix},$$

$$c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}.$$

Proposition 3.3. *For vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{R}^n$ and scalars c, d in $\mathbb{R}$,*

(1) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$,

(2) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$,

(3) $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$,

(4) $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$,

(5) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$,

(6) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$

(7) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$

(8) $1\boldsymbol{u} = \boldsymbol{u}$.

Subtraction can be defined in $\mathbb{R}^n$ by

$$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v}).$$

Linear combinations

Definition 3.4. A vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is called a **linear combination** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ in $\mathbb{R}^n$ if there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k.$$

The set of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ is called the **span** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$, denoted

$$\operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}.$$

Example 3.1. The span of a single nonzero vector in $\mathbb{R}^3$ is a straight line through the origin. For instance,

$$\operatorname{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$$

is a straight line through the origin, having the **parametric form**

$$\begin{cases} x_1 &= & 3t \\ x_2 &= & -t \\ x_3 &= & 2t \end{cases}, \quad t \in \mathbb{R}.$$

Eliminating the parameter t, the parametric equations reduce to two equations about x_1, x_2, x_3,

$$\begin{cases} x_1 & +3x_2 & & = & 0 \\ & 2x_2 & +x_3 & = & 0 \end{cases}$$

Example 3.2. The span of two linearly independent vectors in $\mathbb{R}^3$ is a plane through the origin. For instance,

$$\operatorname{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} : s,\, t \in \mathbb{R} \right\}$$

is a plane through the origin, having the following parametric form

$$\begin{cases} x_1 &= & s & +3t \\ x_2 &= & 2s & -t \\ x_3 &= & -s & +2t \end{cases}$$

Eliminating the parameters s and t, the plane can be described by a single equation

$$3x_1 - 5x_2 - 7x_3 = 0.$$

Example 3.3. Given vectors in $\mathbb{R}^3$,

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

(a) Every vector $\boldsymbol{b} = [b_1, b_2, b_3]^T$ in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

(b) The vector $\boldsymbol{v}_4$ can be expressed as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, and it can be expressed in one and only one way as a linear combination

$$\boldsymbol{v} = 2\boldsymbol{v}_1 - \boldsymbol{v}_2 + 3\boldsymbol{v}_3.$$

(c) The span of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is the whole vector space $\mathbb{R}^3$.

Solution. (a) Let $\boldsymbol{b} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3$. Then the vector equation has the following matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Perform row operations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 1 & 0 & b_2 \\ 0 & 1 & 1 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & -1 & -2 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & b_2 - b_1 \\ 0 & 1 & 2 & 2b_1 - b_2 \\ 0 & 0 & -1 & b_3 + b_2 - 2b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 - b_3 \\ 0 & 1 & 0 & -2b_1 + b_2 + 2b_3 \\ 0 & 0 & 1 & 2b_1 - b_2 - b_3 \end{array}\right]$$

Thus

$$x_1 = b_1 - b_3, \quad x_2 = -2b_1 + b_2 + 2b_3, \quad x_3 = 2b_1 - b_2 - b_3.$$

So $\boldsymbol{b}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

(b) In particular, $\boldsymbol{v}_4 = 2\boldsymbol{v}_1 - \boldsymbol{v}_2 + 3\boldsymbol{v}_3$.

(c) Since $\boldsymbol{b}$ is arbitrary, we have $\operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} = \mathbb{R}^3$.

Example 3.4. Consider vectors in $\mathbb{R}^3$,

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}; \quad \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}, \ \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) The vector $\boldsymbol{u}$ can be expressed as linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ in more than one ways. For instance,

$$\boldsymbol{u} = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 = 4\boldsymbol{v}_1 + 3\boldsymbol{v}_2 = -2\boldsymbol{v}_1 - \boldsymbol{v}_2 + 2\boldsymbol{v}_3.$$

(b) The vector $\boldsymbol{v}$ can not be written as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

Geometric interpretation of vectors

Multiplication of matrices

Definition 3.5. Let A be an $m \times n$ matrix and B an $n \times p$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & a_{n2} & \dots & b_{np} \end{bmatrix}.$$

The **product** (or **multiplication**) of A and B is an $m \times p$ matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}$$

whose (i,k)-entry c_{ik}, where $1 \le i \le m$ and $1 \le k \le p$, is given by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk}.$$

Proposition 3.6. *Let A be an $m \times n$ matrix, whose column vectors are denoted by $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$. Then for any vector $\boldsymbol{v}$ in $\mathbb{R}^n$,*

$$A\boldsymbol{v} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\boldsymbol{a}_1 + v_2\boldsymbol{a}_2 + \dots + v_n\boldsymbol{a}_n.$$

Proof. Write

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Then

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Thus

$$
\begin{aligned}
A\boldsymbol{v} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11}v_1 \\ a_{21}v_1 \\ \vdots \\ a_{m1}v_1 \end{bmatrix} + \begin{bmatrix} a_{12}v_2 \\ a_{22}v_2 \\ \vdots \\ a_{m2}v_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}v_n \\ a_{2n}v_n \\ \vdots \\ a_{mn}v_n \end{bmatrix} \\
&= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\
&= v_1\boldsymbol{a}_1 + v_2\boldsymbol{a}_2 + \cdots + v_n\boldsymbol{a}_n.
\end{aligned}
$$

□

Theorem 3.7. *Let A be an $m \times n$ matrix. Then for any vectors $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^n$ and scalar c,*

(a) $A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$,

(b) $A(c\boldsymbol{u}) = cA\boldsymbol{u}$.

4 The four expressions of a linear system

A general system of linear equations can be written as

$$
\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_1 \\ & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases} \tag{4.1}
$$

We introduce the column vectors:

$$
\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \dots, \quad \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}; \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix};
$$

and the coefficient matrix:

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [\,\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\,].
$$

Then the linear system (4.1) can be expressed by

(a) The **vector equation form**:

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b},$$

(b) The **matrix equation form**:

$$A\boldsymbol{x} = \boldsymbol{b},$$

(c) The **augmented matrix form**:

$$\big[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \,|\, \boldsymbol{b}\big].$$

Theorem 4.1. *The system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is a linear combination of the column vectors of A.*

Theorem 4.2. *Let A be an $m \times n$ matrix. The following statements are equivalent.*

(a) *For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.*

(b) *The column vectors of A span $\mathbb{R}^m$.*

(c) *The matrix A has a pivot position in every row.*

Proof. $(a) \Leftrightarrow (b)$ and $(c) \Rightarrow (a)$ are obvious.

$(a) \Rightarrow (c)$: Suppose A has no pivot position for at least one row; that is,

$$A \overset{\rho_1}{\sim} A_1 \overset{\rho_2}{\sim} \cdots \overset{\rho_{k-1}}{\sim} A_{k-1} \overset{\rho_k}{\sim} A_k,$$

where $\rho_1, \rho_2, \ldots, \rho_k$ are elementary row operations, and A_k is a row echelon matrix. Let $\boldsymbol{e}_m = [0, \ldots, 0, 1]^T$. Clearly, the system $[A \,|\, \boldsymbol{e}_n]$ is inconsistent. Let ρ_i' denote the inverse row operation of ρ_i, $1 \le i \le k$. Then

$$[A_k \,|\, \boldsymbol{e}_m] \overset{\rho_k'}{\sim} [A_{k-1} \,|\, \boldsymbol{b}_1] \overset{\rho_{k-1}'}{\sim} \cdots \overset{\rho_2'}{\sim} [A_1 \,|\, \boldsymbol{b}_{k-1}] \overset{\rho_1'}{\sim} [A \,|\, \boldsymbol{b}_k].$$

Thus, for $\boldsymbol{b} = \boldsymbol{b}_k$, the system $[A \,|\, \boldsymbol{b}]$ has no solution, a contradiction. □

Example 4.1. The following linear system has no solution for some vectors $\boldsymbol{b}$ in $\mathbb{R}^3$.

$$\left\{ \begin{array}{rrrrcl} & 2x_2 & +2x_3 & +3x_4 & = & b_1 \\ 2x_1 & +4x_2 & +6x_3 & +7x_4 & = & b_2 \\ x_1 & +x_2 & +2x_3 & +2x_4 & = & b_3 \end{array} \right.$$

The row echelon matrix of the coefficient matrix for the system is given by

$$\begin{bmatrix} 0 & 2 & 2 & 3 \\ 2 & 4 & 6 & 7 \\ 1 & 1 & 2 & 2 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 4 & 6 & 7 \\ 0 & 2 & 2 & 3 \end{bmatrix} \overset{R_2 - 2R_1}{\sim}$$
$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 2 & 2 & 3 \end{bmatrix} \overset{R_3 - R_2}{\sim} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the following systems have no solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_3 + R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 3 & 0 \\ 0 & 2 & 2 & 3 & 1 \end{array}\right] \overset{R_2 + 2R_1}{\sim}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 2 & 4 & 6 & 7 & 0 \\ 0 & 2 & 2 & 3 & 1 \end{array}\right] \overset{R_3 \leftrightarrow R_1}{\sim} \left[\begin{array}{cccc|c} 0 & 2 & 2 & 3 & 1 \\ 2 & 4 & 6 & 7 & 0 \\ 1 & 1 & 2 & 2 & 0 \end{array}\right].$$

Thus the original system has no solution for $b_1 = 1$, $b_2 = b_3 = 0$.

5 Solution structure of a linear System

Homogeneous system

A linear system is called **homogeneous** if it is in the form $A\boldsymbol{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Note that $\boldsymbol{x} = \mathbf{0}$ is always a solution for a homogeneous system, called the **zero solution** (or **trivial solution**); solutions other than the zero solution $\mathbf{0}$ are called **nontrivial solutions**.

Theorem 5.1. *A homogeneous system $A\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable. Moreover,*

$$\#\{\text{pivot positions}\} + \#\{\text{free variables}\} = \#\{\text{variables}\}.$$

Example 5.1. Find the solution set for the nonhomogeneous linear system

$$\left\{\begin{array}{rrrrrcl} x_1 & -x_2 & & +x_4 & +2x_5 & = & 0 \\ -2x_1 & +2x_2 & -x_3 & -4x_4 & -3x_5 & = & 0 \\ x_1 & -x_2 & +x_3 & +3x_4 & +x_5 & = & 0 \\ -x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & = & 0 \end{array}\right.$$

Solution. Do row operations to reduce the coefficient matrix to the reduced row echelon form:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2 \\ -2 & 2 & -1 & -4 & -3 \\ 1 & -1 & 1 & 3 & 1 \\ -1 & 1 & 1 & 1 & -3 \end{bmatrix} \begin{array}{c} R_2 + 2R_1 \\ R_3 - R_1 \\ \sim \\ R_4 + R_1 \end{array}$$
$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} \begin{array}{c} (-1)R_2 \\ R_3 + R_2 \\ \sim \\ R_4 + R_2 \end{array}$$
$$\begin{bmatrix} (1) & [-1] & 0 & 1 & 2 \\ 0 & 0 & (1) & [2] & [-1] \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the homogeneous system is equivalent to

$$\left\{\begin{array}{rcrrr} x_1 & = & x_2 & -x_4 & -2x_5 \\ x_3 & = & & -2x_4 & +x_5 \end{array}\right.$$

The variables x_2, x_4, x_5 are **free** variables. Set $x_2 = c_1$, $x_4 = c_2$, $x_5 = c_3$. We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 - 2c_3 \\ c_1 \\ -2c_2 + c_3 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Set $x_2 = 1, x_4 = 0, x_5 = 0$, we obtain the basic solution

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set $x_2 = 0, x_4 = 1, x_5 = 0$, we obtain the basic solution

$$\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Set $x_2 = 0, x_4 = 0, x_5 = 1$, we obtain the basic solution

$$\boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The general solution of the system is given by

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3, \quad c_1, c_2, c_3, \in \mathbb{R}.$$

In other words, the solution set is Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

Theorem 5.2. *Let $A\boldsymbol{x} = \boldsymbol{0}$ be a homogeneous system. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions, then the addition and the scalar multiplication*

$$\boldsymbol{u} + \boldsymbol{v}, \quad c\boldsymbol{u}$$

are also solutions. Moreover, any linear combination of solutions for a homogeneous system is again a solution.

Theorem 5.3. *Let $A\boldsymbol{x} = \boldsymbol{0}$ be a homogeneous linear system, where A is an $m \times n$ matrix with p pivot positions. Then system has $n-p$ free variables and $n-p$ basic solutions. The basic solutions can be obtained as follows:* **Setting one free variable equal to 1 and all other free variables equal to 0.**

Nonhomogeneous systems

A linear system $A\boldsymbol{x} = \boldsymbol{b}$ is called **nonhomogeneous** if $\boldsymbol{b} \neq \boldsymbol{0}$. The homogeneous linear system $A\boldsymbol{x} = \boldsymbol{0}$ is called its **corresponding homogeneous linear system**.

Proposition 5.4. *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be solutions of a nonhomogeneous system $A\boldsymbol{x} = \boldsymbol{b}$. Then the difference*

$$\boldsymbol{u} - \boldsymbol{v}$$

is a solution of the corresponding homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$.

Theorem 5.5 (Structure of Solution Set). *Let $\boldsymbol{x}_{\text{nonh}}$ be a solution of a nonhomogeneous system $A\boldsymbol{x} = \boldsymbol{b}$. Let $\boldsymbol{x}_{\text{hom}}$ be the general solutions of the corresponding homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$. Then*

$$\boldsymbol{x} = \boldsymbol{x}_{\text{nonh}} + \boldsymbol{x}_{\text{hom}}$$

are the general solutions of $A\boldsymbol{x} = \boldsymbol{b}$.

Example 5.2. Find the solution set for the nonhomogeneous linear system

$$\left\{ \begin{array}{rrrrrcr} x_1 & -x_2 & & +x_4 & +2x_5 & = & -2 \\ -2x_1 & +2x_2 & -x_3 & -4x_4 & -3x_5 & = & 3 \\ x_1 & -x_2 & +x_3 & +3x_4 & +x_5 & = & -1 \\ -x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & = & 3 \end{array} \right.$$

Solution. Do row operations to reduce the augmented matrix to the reduced row echelon form:

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 1 & 2 & -2 \\ -2 & 2 & -1 & -4 & -3 & 3 \\ 1 & -1 & 1 & 3 & 1 & -1 \\ -1 & 1 & 1 & 1 & -3 & 3 \end{array}\right] \begin{array}{c} R_2+2R_1 \\ R_3-R_1 \\ \sim \\ R_4+R_1 \end{array} \left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 1 & 2 & -2 \\ 0 & 0 & -1 & -2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \begin{array}{c} (-1)R_2 \\ R_3+R_2 \\ \sim \\ R_4+R_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} (1) & [-1] & 0 & 1 & 2 & -2 \\ 0 & 0 & (1) & [2] & [-1] & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then the nonhomogeneous system is equivalent to

$$\left\{\begin{array}{rcrrrr} x_1 & = & -2 & +x_2 & -x_4 & -2x_5 \\ x_3 & = & 1 & & -2x_4 & +x_5 \end{array}\right.$$

The variables x_2, x_4, x_5 are free variables. Set $x_2 = c_1$, $x_4 = c_2$, $x_5 = c_3$. We obtain the general solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2+c_1-c_2-2c_3 \\ c_1 \\ 1-2c_2+c_3 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

or

$$\boldsymbol{x} = \boldsymbol{v} + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3, \quad c_1, c_2, c_3 \in \mathbb{R},$$

called the **parametric form** of the solution set. In particular, setting $x_2 = x_4 = x_5 = 0$, we obtain the **particular solution**

$$\boldsymbol{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The solution set of $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$S = \boldsymbol{v} + \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$$

Example 5.3. The augmented matrix of the nonhomogeneous system

$$\left\{\begin{array}{rrrrrrcr} x_1 & -x_2 & +x_3 & +3x_4 & +3x_5 & +4x_6 & = & 2 \\ -2x_1 & +2x_2 & -x_3 & -3x_4 & -4x_5 & -5x_6 & = & -1 \\ -x_1 & +x_2 & & & -x_5 & -4x_6 & = & -5 \\ -x_1 & +x_2 & +x_3 & +3x_4 & +x_5 & -x_6 & = & -2 \end{array}\right.$$

has the reduced row echelon form

$$\left[\begin{array}{cccccc|c} (1) & [-1] & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & (1) & [3] & [2] & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & (1) & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Then a particular solution $\boldsymbol{x}_p$ for the system and the basic solutions of the corresponding homogeneous system are given by

$$\boldsymbol{x}_p = \begin{bmatrix} -3 \\ 0 \\ -3 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The general solution of the system is given by

$$\boldsymbol{x} = \boldsymbol{x}_p + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3.$$

Example 5.4. For what values of a and b the linear system

$$\begin{cases} x_1 & +2x_2 & +ax_3 & = & 0 \\ 2x_1 & +bx_2 & & = & 0 \\ 3x_1 & +2x_2 & +x_3 & = & 0 \end{cases}$$

has nontrivial solutions. Answer: $8a + (3a-1)(b-4) = 0$.

6 Linear dependence and independence

Lecture 6

Definition 6.1. Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ in $\mathbb{R}^n$ are said to be **linearly independent** provided that, whenever

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$$

for some scalars $c_1, c_2, \ldots, c_k$, then $c_1 = c_2 = \cdots = c_k = 0$.

The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are called **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}.$$

Example 6.1. The vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$ are linearly independent.

Solution. Consider the linear system

$$x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The system has the only zero solution $x_1 = x_2 = x_3 = 0$. Thus $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent.

Example 6.2. The vector $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$ in $\mathbb{R}^3$ are linearly dependent.

Solution. Consider the linear system

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The system has one free variable. There is nonzero solution. Thus $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly dependent.

Example 6.3. The vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$ in $\mathbb{R}^3$ are linearly dependent.

Theorem 6.2. *The basic solutions of any homogeneous linear system are linearly independent.*

Proof. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ be basic solutions of a linear system $A\boldsymbol{x} = \mathbf{0}$, corresponding to free variables $x_{j_1}, x_{j_2}, \ldots, x_{j_k}$. Consider the linear combination

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \mathbf{0}.$$

Note that the j_i coordinate of $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$ is c_{j_i}, $1 \le i \le k$. It follows that $c_{j_i} = 0$ for all $1 \le i \le k$. This means that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are linearly independent. □

Theorem 6.3. *Any set of vectors containing the zero vector* $\mathbf{0}$ *is linearly dependent.*

Theorem 6.4. *Let* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$ *be vectors in* $\mathbb{R}^n$*. If* $p > n$*, then* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$ *are linearly dependent.*

Proof. Let $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p]$. Then A is an $n \times p$ matrix, and the equation $A\boldsymbol{x} = \mathbf{0}$ has n equations in p unknowns. Recall that for the matrix A the number of pivot positions plus the number of free variables is equal to p, and the number of pivot positions is at most n. Thus, if $p > n$, there must be some free variables. Hence $A\boldsymbol{x} = \mathbf{0}$ has nontrivial solutions. This means that the column vectors of A are linearly dependent. □

Theorem 6.5. *Let* $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ *be a set of vectors in* $\mathbb{R}^n$*,* $(p \ge 2)$*. Then* S *is linearly dependent if and only if one of vectors in* S *is a linear combination of the other vectors.*

Moreover, if S *is linearly dependent and* $\boldsymbol{v}_1 \ne \mathbf{0}$*, then there is a vector* $\boldsymbol{v}_j$ *with* $j \ge 2$ *such that* $\boldsymbol{v}_j$ *is a linear combination of the preceding vectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{j-1}$*.*

Note 2. The condition $\boldsymbol{v}_1 \ne \mathbf{0}$ in the above theorem can not be deleted. For example, the set

$$\left\{ \boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent. But $\boldsymbol{v}_2$ is not a linear combination of $\boldsymbol{v}_1$.

Theorem 6.6. *The column vectors of a matrix* A *are linearly independent if and only if the linear system*

$$A\boldsymbol{x} = \mathbf{0}$$

has the only zero solution.

Proof. Let $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n]$. Then the linear system $A\boldsymbol{x} = \mathbf{0}$ is the vector equation

$$\boldsymbol{x}_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}.$$

Then $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are linear independent is equivalent to that the system has only the zero solution. □