Math113: Final Exam, Spring 2006

Name:____________________

ID No.____________________ Lecture Section:____________________

Problem	1 (10 pts)	2 (13 pts)	3 (14 pts)	4 (14 pts)	5 (14 pts)	6 (14 pts)	7 (21 pts)	Total (100 pts)
Score								

1. (10 pts) Show that for all real numbers $\alpha, \beta, \gamma, \delta$ we always have

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0.$$

Solution: Since $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we have

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\beta\cos\delta - \sin\beta\sin\delta \end{vmatrix} \begin{matrix} C_3 + (\sin\delta)C_1 \\ \sim \\ C_3 - (\cos\delta)C_2 \end{matrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} = 0.$$

2. (13 pts) Let $\mathbf{P}_3$ be the vector space of all real polynomials in one variable t with degree at most 3. Let $S = \{f_1, f_2, f_3, f_4, f_5\}$, where

$$\begin{aligned} f_1(t) &= -2 + t + 3t^2 + t^3, \\ f_2(t) &= -5 + 3t + 11t^2 + 7t^3, \\ f_3(t) &= 8 - 5t - 19t^2 - 13t^3, \\ f_4(t) &= t + 7t^2 + 5t^3, \\ f_5(t) &= -17 + 5t + t^2 - 3t^3. \end{aligned}$$

(a) Find a subset $\mathcal{B}$ of S so that $\mathcal{B}$ is a basis of Span (S).

(b) Express each polynomial in S as a linear combination of the polynomials in $\mathcal{B}$.

(c) Is Span $(S) = \mathbf{P}_3$? Justify your statement.

Solution. (a) Since $\{1, t, t^2, t^3\}$ is basis of $\mathbf{P}_3$, the coordinate vectors of the polynomials $f_1(t), f_2(t), \ldots, f_5(t)$ are columns of the matrix

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \\ \sim \\ R_3 - 3R_1 \\ R_4 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \sim \\ R_3 - 2R_2 \\ R_4 - 4R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & -5 & 26 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix} \begin{matrix} (-1/4)R_4 \\ \sim \\ R_3 \leftrightarrow R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -5 & 26 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (-1/4)R_4 \\ \sim \\ R_3 \leftrightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the set $\{f_1(t), f_2(t), f_4(t)\}$ form a basis of Span (S).

(b) Thus $f_1(t) = f_1(t)$, $f_2(t) = f_2(t)$, $f_4(t) = f_4(t)$, and

$$f_3(t) = f_1(t) - 2f_2(t), \qquad f_5(t) = f_1(t) + 3f_2(t) - 5f_4(t).$$

(c) No. The space $\mathbf{P}_3$ is of dimension 4 and Span (S) is of dimension 3.

3. (14 pts) Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) (9 pts) Determine which of the two matrices are diagonalizable. Justify your answer.

(b) (5 pts) For the matrix which is diagonalizable, compute its 100th power.

Solution. (a) The matrix A has only one eigenvalue $\lambda = 2$, whose eigenspace has dimension one. Then A is not diagonalizable. The matrix B has eigenvalues $\lambda = 2, 1$.

For $\lambda = 2$, the system

$$\begin{bmatrix} 0 & 4 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has basic solutions $v_1 = [1, 0, 0]^T$, $v_2 = [0, 0, 1]^T$.

For $\lambda = 1$, the system

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has basic solution $v_3 = [4, -1, 1]^T$. The eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$. So B is diagonalizable.

(b) Let $P = [v_1, v_2, v_3]$. Then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right] \sim$$

$$A^{100} = P\text{Diag}[2, 2, 1]^{100} P^{-1} = \begin{bmatrix} 2^{100} & 4 \cdot 2^{100} - 4 & 0 \\ 0 & 1 & 0 \\ 0 & 2^{100} - 1 & 2^{100} \end{bmatrix}.$$

4. (14 pts) Let $\mathbf{M}_{2\times 2}$ be the vector space of all 2×2 real matrices. Note that the set

$$\mathcal{B} = \left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of $\mathbf{M}_{2\times 2}$. The coordinate mapping relative to the basis $\mathcal{B}$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

is a one-to-one linear transformation from $\mathbf{M}_{2\times 2}$ onto $\mathbb{R}^4$. Note that every linear algebra problem for $\mathbf{M}_{2\times 2}$ can be interpreted as a corresponding problem for $\mathbb{R}^4$ via this coordinate mapping.

Let $T : \mathbf{M}_{2\times 2} \to \mathbf{M}_{2\times 2}$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 - x_4 & 2x_1 + x_2 + x_3 - x_4 \\ x_2 - x_3 + x_4 & x_1 + 2x_2 - x_3 - x_4 \end{bmatrix}.$$

(a) Write down the linear transformation $S : \mathbb{R}^4 \to \mathbb{R}^4$ corresponding to the linear transformation $T : \mathbf{M}_{2\times 2} \to \mathbf{M}_{2\times 2}$ via the above coordinate mapping; and find the standard matrix A of S.

(b) Find a basis for the kernel of T.

(c) Find a basis for the range of T. (Note: the range of T is the set consisting of all matrices $T(A)$, where $A \in \mathbf{M}_{2\times 2}$.)

Solution. (a) The linear transformation S is given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) == \begin{bmatrix} x_1 + x_3 - x_4 \\ 2x_1 + x_2 + x_3 - x_4 \\ x_2 - x_3 + x_4 \\ x_1 + 2x_2 - x_3 - x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A\boldsymbol{x}.$$

(b) Perform row operations:

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ \sim \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 0 \end{bmatrix} \begin{matrix} R_3 - R_2 \\ \sim \\ R_4 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{matrix} R_1 - (1/2)R_4 \\ R_2 + (1/2)R_4 \\ \sim \\ R_3 \leftrightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $A\boldsymbol{x} = \mathbf{0}$ has one basic solution $\boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. Thus the kernel of T has a basis $\left\{ \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.

(c) The column space $\operatorname{Col} A$ has a basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$. Hence, the range of T has a basis $\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \right\}$.

5. (14 pts) Let W be the solution set of the homogeneous system

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 0 \\ -2x_1 + x_2 - 3x_3 - x_4 &= 0 \end{aligned}$$

(a) Solve the linear system and express the general solution in **parametric vector form**.

(b) Find an orthogonal basis for W.

(c) Let $\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ be a vector in $\mathbb{R}^4$. Find a vector $\boldsymbol{v}$ in W that satisfies the condition:

$$||\boldsymbol{y} - \boldsymbol{v}|| < ||\boldsymbol{y} - \boldsymbol{w}|| \quad \text{for every } \boldsymbol{w} \text{ in } W \text{ distinct from } \boldsymbol{v}.$$

Solution. (a) Perform row operations:

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & -5 \end{bmatrix}.$$

The general solution is given by

$$\boldsymbol{x} = s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

(b) The subspace W has an orthogonal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, where

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 1 \end{bmatrix} - \frac{-2+5}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 1 \end{bmatrix}.$$

(c)

$$\begin{aligned} \boldsymbol{v} &= \text{Proj}_W \boldsymbol{y} = \frac{-1+2+3}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3+8-3+4}{27} \begin{bmatrix} 3 \\ 4 \\ -1 \\ 1 \end{bmatrix} \\ &= \frac{4}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{9} \begin{bmatrix} 3 \\ 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 28/9 \\ 8/9 \\ 4/9 \end{bmatrix}. \end{aligned}$$

6. (14 pts) Given that the following 4 vectors form a basis of $\mathbb{R}^4$:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \ \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}, \ \boldsymbol{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt orthogonalization process to $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ to obtain an orthogonal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ for $\mathbb{R}^4$.

(b) Let $\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Express $\boldsymbol{y}$ as a linear combination of the orthogonal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ obtained in part (a).

Solution. (a)

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}; \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{2+4}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix};$$

$$\boldsymbol{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} - \frac{1-4+3}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1-3+2}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix};$$

$$\begin{aligned} \boldsymbol{u}_4 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1+2+1}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1-1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1-2+3}{18} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/9 \\ -1/9 \\ 0 \\ -2/9 \end{bmatrix}. \end{aligned}$$

(b)

$$\begin{aligned} \boldsymbol{y} &= \frac{1+4+3}{6}\boldsymbol{u}_1 + \frac{1-3+4}{3}\boldsymbol{u}_2 + \frac{1-4+9+8}{18}\boldsymbol{u}_3 + \frac{\frac{2-2-8}{9}}{\frac{4+1+4}{81}}\boldsymbol{u}_4 \\ &= \frac{4}{3}\boldsymbol{u}_1 + \frac{2}{3}\boldsymbol{u}_2 + \frac{7}{9}\boldsymbol{u}_3 - 8\boldsymbol{u}_4. \end{aligned}$$

7. (21 pts) For each of the following problems, circle only one answer. Correct answer: 3 ponts. Wrong answer: -1 point. No answer: 0 point.

(1) Let $A\boldsymbol{x} = \mathbf{0}$ be a linear system in 100 variables and 90 equations. If $\dim \operatorname{Nul} A = 20$, then the dimension of $\operatorname{Col} A$ is equal to

(a) 50;
(b) 60;
(c) 70;
✓(d) 80;
(e) 90;
(f) 100.

(2) Let A be an $m \times n$ matrix such that the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ in $\mathbb{R}^m$. Which of the following statements about A is NOT valid?

(a) $m \leq n$;
(b) The columns of A span $\mathbb{R}^m$;
(c) Each row of A has a pivot position;
✓(d) Every column of A has a pivot position.

(3) Let A and B be $m \times n$ matrices such that A is row-equivalent to B. Which of the following statements is INCORRECT?

(a) There exists an $m \times m$ invertible matrix P such that $A = PB$;
✓(b) There exists an $n \times n$ invertible matrix Q such that $A = BQ$;
(c) $\operatorname{Row} A = \operatorname{Row} B$;
(d) $\operatorname{rank} A = \operatorname{rank} B$.

(4) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a one-to-one linear transformation with the standard matrix A. Which of the following statements is CORRECT?

(a) $\operatorname{Col} A = \mathbb{R}^m$;
✓(b) $\operatorname{Nul} A$ is the zero subspace of $\mathbb{R}^n$;
(c) $m < n$;
(d) The range of T is $\mathbb{R}^m$.

(5) Let B be the reduced row echelon form of an $m \times n$ matrix A. Which of the following statements is TRUE?

(a) The pivot columns of A form a basis of $\operatorname{Col} B$;
(b) The pivot columns of B form a basis of $\operatorname{Col} A$;
(c) The non-zero rows of A form a basis of $\operatorname{Row} B$;
✓(d) The non-zero rows of B form a basis of $\operatorname{Row} A$.

(6) Let $B = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4]$ be a 4×4 invertible matrix. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that $T(\boldsymbol{b}_i) = \boldsymbol{e}_i$, $i = 1, 2, 3, 4$, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the standard basis of $\mathbb{R}^4$. Let A denote the standard matrix of T. Then the $(1, 2)$-entry of A is equal to

(a) the $(1, 2)$-entry of B;
✓(b) the $(1, 2)$-entry of B^{-1};
(c) the $(2, 1)$-entry of B;
(d) the $(2, 1)$-entry of B^{-1}.

(7) Let A be a 3×3 square matrix. Which of the following conditions CANNOT guarantee that A is diagonalizable?

(a) A has two distinct eigenspaces E_1, E_2, and $\dim E_1 + \dim E_2 = 3$.
(b) A is upper triangular with $1, 2, 3$ being its main diagonal entries;
(c) A has 3 distinct eigenvalues;
✓(d) A has 3 distinct eigenvectors.