Math113: Final Exam, Fall 2004

Name:_____   Tutor's Name:_____

ID No._____   Section:_____

Problem	1 (15)	2 (15)	3 (15)	4 (10)	5 (15)	6 (5)	7 (15)	8 (10)	Total (100)
Score									

1. (15 pts)

   (a) Solve the system of linear equations:

   $$\begin{cases} & & x_3 & +2x_4 & +3x_5 & = & 0 \\ x_1 & +2x_2 & +2x_3 & +2x_4 & +3x_5 & = & 0 \\ 2x_1 & +4x_2 & +x_3 & -2x_4 & -3x_5 & = & 0 \\ 2x_1 & +4x_2 & +3x_3 & +2x_4 & +3x_5 & = & 0 \end{cases} \tag{1}$$

   (b) Find a basis for the orthogonal complement of the solution space of the linear system (1) in $\mathbb{R}^5$.

   (c) Is there any linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ such that the range of T is the solution space of the linear system (1)? If yes, find the standard matrix of such a linear transformation T.

2. (15 pts) Let $\mathbf{P}_3$ be the vector space of all polynomials in one variable t of degree at most 3. Note that $\mathcal{B} = \{1, t, t^2, t^3\}$ is a basis of $\mathbf{P}_3$. Let

$$
\begin{aligned}
f_1(t) &= 1 + t + 2t^2 + 3t^3, \\
f_2(t) &= 2 + 2t + 4t^2 + 6t^3, \\
f_3(t) &= 1 + 4t^2 + 6t^3, \\
f_4(t) &= 3 + 5t + 2t^2 + 3t^3, \\
f_5(t) &= 2 + 3t + 3t^2 + 3t^3.
\end{aligned}
$$

(a) Find the coordinate vectors for $f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)$ relative to the basis $\mathcal{B}$ respectively.

(b) Find a basis for $\text{Span}\{f_1, f_2, f_3, f_4, f_5\}$ from the set $\{f_1, f_2, f_3, f_4, f_5\}$ by row operations.

(c) Determine the dimension of $\text{Span}\{f_1, f_2, f_3, f_4, f_5\}$.

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3. (15 pts) Let $A = \begin{bmatrix} 3a & 4 & 2 \\ -6 & 2a & 3 \\ 12 & 2 & 1 \end{bmatrix}$, where a is an unspecified parameter.

(a) Calculate $\det A$.

(b) Find all values for a so that the the matrix A has zero as an eigenvalue.

(c) Find all values for a so that the rank of A is 3.

4. (10 pts) Let $\mathcal{B} = \{v_1, v_2, v_3\}$ and $\mathcal{C} = \{w_1, w_2, w_3\}$ be two bases for a subspace W of $\mathbb{R}^4$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}; \quad w_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, w_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}.$$

(a) Let v be a vector in W whose coordinate vector relative to the basis $\mathcal{C}$ is $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. Find the coordinate vector of v relative to the basis $\mathcal{B}$.

(b) (Not for students in Prof. Qian's Lecture.) If a vector w in W has the coordinate vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ relative to the basis $\mathcal{C}$, find its coordinate vector relative to the basis $\mathcal{B}$ in terms of a, b, c.

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5. (15 pts) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Is A diagonalizable? If yes, find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

(b) Compute the matrix A^{100}.

(c) Let Q be an invertible 3×3 matrix. Is the matrix $B = QAQ^{-1}$ diagonalizable? If yes, find a matrix U such that $U^{-1}BU$ is a diagonal matrix.

6. (5 pts) Let A be a 3×3 matrix having distinct eigenvalues $1, -1, 0$. Show that $A^3 = A$.

7. (15 pts)

(a) Let W be a subspace of $\mathbb{R}^4$ defined by the linear system $A\boldsymbol{x} = \boldsymbol{0}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Find the shortest distance from the point $(1, 2, 2, -5)$ to W.

(b) Find the standard matrix of the orthogonal projection $\mathrm{Proj}_W : \mathbb{R}^4 \to \mathbb{R}^4$.

$A =$

8. (10 pts) Let W be a subspace of $\mathbb{R}^4$ with a basis $\mathcal{B} = \{a_1, a_2, a_3\}$, where

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 0 \\ 6 \\ 3 \\ -3 \end{bmatrix}.$$

(a) Find an orthogonal basis for W by the Gram-Schmidt process.

(b) Find a QR-decomposition of the matrix A whose column vectors are a_1, a_2, a_3.