# MATH 113 Introduction to Linear Algebra – Homework Set 4

§5.1: 6, 8, 16; §5.2: 12, 18; §5.3: 6, 16, 24.
§6.1: 16, 24, 28; §6.2: 6, 10, 20; §6.3: 2, 10, 14; §6.4: 6, 12.

1. Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

1) Diagonalize $A$;
2) Use the diagonalization to find $A^{10}$;
3) Determine the invertibility of $A$ from its diagonalization.
4) Use the diagonalization to compute $A^5 - 2A^4 + A^3$.

2. The city of Clear Water Bay maintains a constant population of 30,000. A political science study estimated that there are 15,000 independents, 9,000 democrats, and 6,000 liberals. It was also estimated that each year 20% of democrats and 10% of liberals become independents, 20% of independents and 10% of liberals become democrats, 10% of independents and 10% of democrats become liberals. Find the political composition of Clear Water Bay in the long run.

3. Determine whether the following statements are true or false (no reason needed).
1) If all eigenvalues of $A$ are 0, then $A = 0$;
2) If $A^9 = 0$, then all eigenvalues of $A$ vanish;
3) If $v$ is an eigenvector of $A$, then $v$ is an eigenvector of $A^2$;
4) If $v$ is an eigenvector of $A$, then $v$ is an eigenvector of $A^T$;
5) If $u$ and $v$ are eigenvectors of $A$, then $u + v$ is also an eigenvector of $A$;
6) If $u$ is an eigenvector of $A$, then $2u$ is an eigenvector of $A$;
7) If $A$ and $B$ are diagonalizable, then $AB$ is diagonalizable;
8) If an invertible matrix $A$ is diagonalizable, then $A^{-1}$ is also diagonalizable;
9) Similar matrices have the same eigenvalues;
10) If two matrices have the same eigenvalues, then they are similar.

4. Consider

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1) Find an orthogonal basis of Col$A$;
2) Find the orthogonal projection of $u$ on Col$A$;
3) Find the distance from $u$ to Col$A$.

5. Find numbers $a$, $b$, $c$, such that

$$U = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix},$$

is an orthogonal matrix. Then find $U^{-1}$.

6. Determine whether the following statements are true or false (no reason needed).

1) If $a, b, c, d$ are vectors in $R^n$, $a$ is orthogonal to $c$ and $b$ is orthogonal to $d$, then $a + b$ is orthogonal to $c + d$.

2) If $A$ is an invertible $n \times n$ matrix, then the column vectors of $A$ form an orthogonal basis.

3) If $\{v_1, v_2, v_3\}$ is an orthonormal set, then it is linearly independent.

4) If $\{u, v, w\}$ is an orthogonal basis of $\mathbf{R}^3$, then $\{2u, -3v, 5w\}$ is also an orthogonal basis.

5) If $u \perp v$, $v \perp w$, then $u \perp w$.

6) If $u \in W$ and $u \in W^{\perp}$, then $u = 0$.

7) If the orthogonal projection of $y$ onto $W$ is $\hat{y}$, then the orthogonal projection of $10y$ onto $W$ is $10\hat{y}$.

8) If column vectors of a matrix is orthogonal, then the row vectors of the matrix is also orthogonal.

9) If $U$ and $V$ are orthogonal matrices, then $U + V$ is an orthogonal matrix.

10) If $U$ and $V$ are orthogonal matrices, then $UV^{-1}$ is an orthogonal matrix.