Counting

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1 Counting Principle

Let X, Y be finite sets. If X, Y are disjoint, then

$$|X \cup Y| = |X| + |Y|.$$

For two tasks T_1 and T_2 to be performed in sequence, if the task T_1 can be performed in m ways, and for each of these m ways the task T_2 can be performed in n ways, then the task sequence T_1T_2 can be performed in mn ways. Using set notation, let X be the set of ways to perform the task T_1 and Y the set of ways to perform the task T_2, then the product

$$X \times Y = \{(x, y) \,|\, x \in X, y \in Y\}$$

is the set of ways to perform the task sequence T_1T_2, and

$$|X \times Y| = |X||Y|.$$

Example 1.1. Suppose a lady has three hats, seven shirts, five skirts, and four pairs of shoes. Assume all hats, shirts, skirts, and shoes are distinct. In how many ways can the lady dress herself by wearing one hat, one shirt, one skirt, and one pair of shoes?

$$\text{answer} = 3 \cdot 7 \cdot 5 \cdot 4 = 420.$$

Example 1.2. Math courses *Calculus*, *Linear Algebra*, and *Discrete Mathematics* are taught by twenty, fifteen, and ten different instructors respectively in Mega University. In how many ways can a student take two of the three courses by selecting instructors?

$$\text{Answer} = 20 \cdot 15 + 20 \cdot 10 + 15 \cdot 10 = 650.$$

Let X and Y be finite sets. Let $f : X \to Y$ be a surjective function. If the inverse image

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}$$

has equal k elements for each $y \in Y$, then

$$|X| = | \cup_{y \in Y} f^{-1}(y)| = \cup_{y \in Y} |f^{-1}(y)| = k|Y| \quad \text{or} \quad |Y| = \frac{|X|}{k}.$$

2 Permutations

Let A be a set of n objects. An arrangement of r elements from A in linear order is called an r-**permutation** of n objects. Let $P(A, r)$ denote the set of all r-permutations of A. The number of r-permutations of n objects is denoted by

$$P_r^n = |P(A, r)|.$$

In the process of producing an r-permutation of n objects, the 1st element can be selected in n choices, the 2nd in $n - 1$ choices, the 3rd in $n - 2$ choices, and so on. Thus

$$P_r^n = n(n-1)\cdots(n-r+1).$$

In particular, when $r = n$, an n-permutation of n objects is simply called a **permutation** of n objects. The number of permutations of n objects is

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1,$$

where $n!$ is read "n factorial".

Example 2.1. How many possible seating plans can be made for four people to be seated at a round table?

Solution. Let $A = \{a, b, c, d\}$ be the set of four persons. Let X be the set of all permutations of A, and Y the set of all round permutations of A.

We define a map $f : X \to Y$ as follows: for each permutation $x_1x_2x_3x_4$, $f(x_1x_2x_3x_4)$ is the round permutation by joining the front side of x_1 to the end side of x_4, and $x_1x_2x_3x_4x_1\cdots$ forms a clockwise direction of $f(x_1x_2x_3x_4)$.

Clearly, f is onto. This is because for each round permutation, we may separate any two neighbors and stretch them into a linear permutation. There are exactly 4 places to separate a round permutation into a linear permutation. For instance, the four cyclic permutations

$$abcd, \quad bcda, \quad cdab, \quad dabc$$

are sent to the same round permutation $abcda\cdots$ in clockwise direction. We then have

$$|X| = 4|Y|.$$

Thus

$$|Y| = \frac{|X|}{4} = \frac{4!}{4} = 3! = 6.$$

Proposition 2.1. *The number of round permutations of n distinct objects is*

$$\frac{n!}{n} = (n-1)!.$$

Corollary 2.2. *The number of necklaces with n (≥ 3) distinct beads is*

$$\frac{(n-1)!}{2}.$$

Note that elements of a set are always distinct. When considering indistinguishable elements, we need the concept of multisets. By a **multiset** we mean a collection of objects such that some of them may be identically same, called **indistinguishable**. For instance,

$$\{a, b, b, c, c, c, d, e, e\}$$

is a multiset of 9 objects; it is *not* a set. The following

$$\{2, 2, 3, 4, 4, 4, 4, 5, 5, 5\}$$

is a multiset of 10 objects; and it is *not* a set.

Let A be a multiset of n objects of k distinguishable types. If there are n_i indistinguishable objects for the ith type, $i = 1, 2, \ldots, k$, we call A a **multiset of type** $(n_1, n_2, \ldots, n_k)$.

Example 2.2. In how many ways can 6 color balls of same size, of which 2 are white, 3 are black, and 1 is red, be arranged in linear order?

Solution. We denote 6 balls by letters w, w, b, b, b, r. To make the balls distinguishable, we label the balls of the same color with numbers. Then we have 6 distinct balls

$$w_1, w_2, b_1, b_2, b_3, r_1.$$

Let X be the set of permutations of the set $\{w_1, w_2, b_1, b_2, b_3, r_1\}$, and Y the set of permutations of the multiset $\{w, w, b, b, b, r\}$. Let $f : X \to Y$ be the map such that each permutation of $\{w_1, w_2, b_1, b_2, b_3, r_1\}$ is sent to a permutation of $\{w, w, b, b, b, r\}$ by merely erasing the labels of balls in the permutation. For instance,

$$12 = 2!3!1! \left\{ \begin{array}{c} b_1w_1b_2b_3r_1w_2 \\ b_1w_1b_3b_2r_1w_2 \\ b_2w_1b_1b_3r_1w_2 \\ b_2w_1b_3b_1r_1w_2 \\ b_3w_1b_1b_2r_1w_2 \\ b_3w_1b_2b_1r_1w_2 \\ b_1w_2b_2b_3r_1w_1 \\ b_1w_2b_3b_2r_1w_1 \\ b_2w_2b_1b_3r_1w_1 \\ b_2w_2b_3b_1r_1w_1 \\ b_3w_2b_1b_2r_1w_1 \\ b_3w_2b_2b_1r_1w_1 \end{array} \right\} \stackrel{f}{\longmapsto} bwbbrw.$$

Clearly, f is onto. For each permutation π of the multiset $\{w, w, b, b, b, r\}$, the inverse image $f^{-1}(\pi)$ consists of $2!3!1!$ permutations of the set $\{w_1, w_2, b_1, b_2, b_3, r_1\}$. Thus

$$|X| = 2!3!1!|Y|.$$

Therefore

$$|Y| = \frac{|X|}{2!3!1!} = \frac{6!}{2!3!1!} = 60.$$

Theorem 2.3. *The number of permutations of* n *objects of type* $(n_1, \ldots, n_k)$, *where* $n = n_1 + \cdots + n_k$, *is given by*

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Example 2.3. How many ways can five same calculus books, three same physics books, and two same chemistry books be arranged in a bookshelf?

$$\text{Answer} = \frac{10!}{5!3!2!} = 2520.$$

Corollary 2.4. *The number of sequences of* 0 *and* 1 *of length* n *having exactly* r **ones** *and* $(n-r)$ **zeros** *is given by*

$$\frac{n!}{r!(n-r)!}.$$

Example 2.4. Counting the number of nondecreasing lattice paths from the origin (0,0) to the point (6,4).

Solution. Each such lattice path can be viewed as a walk by moving 1 unit length to the right or up. Let us denote the move of 1 unit to the right by R, and the move of 1 unit up by U. Then each such lattice path can be viewed as a sequence of R and U of length 10 having exactly 6 R's and 4 U's. For instance,

$$RRURRRUURU, \quad URRRUURRUR, \quad RUURRUURRR.$$

Thus

$$\text{answer} = \frac{10!}{6!4!} = 210.$$

Proposition 2.5. *The number of nondecreasing lattice paths from* $(0,0)$ *to* (a,b) *with* $a, b \in \mathbb{N}$, *is given by*

$$\frac{(a+b)!}{a!b!}.$$

Thinking Problem. Find a formula for the number of round permutations of n objects of type $(n_1, n_2, \ldots, n_k)$. (Hint: Applying the Möbius inversion formula and the Euler ϕ function; see Problem set. Not required.)

3 Combination

A **combination** is a collection of objects (order is immaterial) from a given source of objects. An r**-combination** of n objects is a collection of r objects from a source of n distinct objects, i.e., an r-subset of an n-set. We denote by $\mathcal{P}_r(A)$ or $\binom{A}{r}$ the set of all r-subsets of A. The number of r-combinations n objects is denoted by

$$\binom{n}{r} := |\mathcal{P}_r(A)| = \#\binom{A}{r},$$

read "n choose r". Other common notations for $\binom{n}{r}$ are $C(n,r), nCr, C_n^k, C_k^n$.

Example 3.1. Find the number of 3-subsets of a 5-set

$$A = \{a,b,c,d,e\}$$

Solution. First Method: Let X be the set of permutations of A, and Y the set of 3-subsets of A. Consider the map $f : X \to Y$, defined by

$$f(x_1x_2x_3x_4x_5) = \{x_1,x_2,x_3\}, \quad x_1x_2x_3x_4x_5 \in X.$$

Clearly, f is onto. For each $S \in Y$, i.e., a 3-subset $S \subset A$, there are $3!2!$ permutations π of A such that $f(\pi) = S$. For instance, for $S = \{a,c,e\}$, we have

$$12 = 3!2! \left\{ \begin{array}{c} acebd \\ aecbd \\ caebd \\ ceabd \\ eacbd \\ ecabd \\ acedb \\ aecdb \\ caedb \\ ceadb \\ eacdb \\ ecadb \end{array} \right\} \stackrel{f}{\longmapsto} \{a,c,e\}.$$

Thus $|X| = 3!2!|Y|$. Therefore,

$$\binom{5}{3} = |Y| = \frac{|X|}{3!2!} = \frac{5!}{3!2!} = 10.$$

Second Method: Let X be the set of 3-permutations of A and Y the set of 3-subsets of A. Let $f : X \to Y$ be defined by

$$f(x_1x_2x_3) = \{x_1, x_2, x_3\}, \quad x_1x_2x_3 \in X.$$

Clearly, f is onto. The 3! permutations of $\{x_1, x_2, x_3\}$ are sent to $\{x_1, x_2, x_3\}$. Thus $|X| = 3!|Y|$. Therefore

$$|Y| = \frac{|X|}{3!} = \frac{P_3^5}{3!} = \frac{5 \cdot 4 \cdot 3}{3!} = 10.$$

Theorem 3.1. *The number of r-combinations of n objects is*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{P_r^n}{r!}.$$

Theorem 3.2. (Binomial Theorem and Expansion)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof.

$$\begin{aligned}
(x+y)^n &= \underbrace{(x+y)(x+y)\cdots(x+y)}_{n} \\
&= \sum u_1u_2\cdots u_n \quad (u_i = x \text{ or } y,\ 1 \le i \le n) \\
&= \sum_{k=0}^{n} \left\{ \begin{array}{l} \#\text{ of sequences of } x \;\&\; y \text{ of length} \\ n \text{ with exact } k\ x\text{'s} \;\&\; (n-k)\ y\text{'s} \end{array} \right\} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\end{aligned}$$

□

Example 3.2. A department consisting of 30 faculty members. Find the number of ways to form a UG committee, a PG committee, and a promotion committee consisting of 5, 3, and 4 respectively. I how many ways can the three committees be formed without any restriction?

Solution.

$$\text{answer} = \binom{30}{5}\binom{30}{3}\binom{30}{4}.$$

A collection of ordered k disjoint subsets of an n-set is called a **combination** of n objects of type $(n_1, n_2, \ldots, n_k)$ if the k subsets have the cardinalities $n_1, n_2, \ldots, n_k$ and

$$n = n_1 + n_2 + \cdots + n_k.$$

A combination of n objects of type $(n_1, n_2, \ldots, n_k)$ can be viewed as a placement of n objects into k boxes so that the 1st box contains n_1 objects, the 2nd box contains n_2 objects, ..., and the kth box contains n_k objects. We denote by

$$\binom{n}{n_1, n_2, \ldots, n_k}$$

the number of combinations of n objects of type $(n_1, n_2, \ldots, n_k)$, read "n choose n_1, n_2, dot dot dot, and n_k".

Example 3.3. How many ways can six distinct objects be placed into three distinct boxes so that the 1st, 2nd, and 3rd boxes contain 2, 3, and 1 objects respectively?

Solution. Let $A = \{a, b, c, d, e, f\}$ be the set of six objects. Let X be the set of permutations of A, and Y the set of placements of elements of A into three boxes so that the 1st, 2nd, and 3rd boxes receives 2, 3, and 1 elements, respectively. There is a map $f : X \to Y$, defined by

$$f(x_1x_2x_3x_4x_5x_6) = \{x_1, x_2\}\{x_3, x_4, x_5\}\{x_6\},$$

where $x_1x_2x_3x_4x_5x_6$ is a permutation of A.

Clearly, f is onto. For each $P \in Y$, there are $2!3!1!$ permutations of A sent to P. For instance, for $P = \{a, c\}\{b, d, f\}\{e\}$,

$$2!3!1! \left\{ \begin{array}{c} acbdfe \\ acbfde \\ acdbfe \\ acdfbe \\ acfbde \\ acfdbe \\ cabdfe \\ cabfde \\ cadbfe \\ cadfbe \\ cafbde \\ cafdbe \end{array} \right\} \overset{f}{\mapsto} \{a, c\}\{b, d, f\}\{e\}$$

Thus $|X| = 2!3!1!|Y|$. Therefore

$$|Y| = \frac{|X|}{2!3!1!} = \frac{6!}{2!3!1!} = \binom{6}{2, 3, 1} = 60.$$

Theorem 3.3. *The number of ways to place n distinct objects into k distinct boxes, so that the 1st, 2nd, …, kth boxes contain $n_1, n_2, \ldots, n_k$ objects respectively, equals*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2! \cdots n_k!}.$$

Remark. When considering placement of n objects into two boxes of type $(r, n-r)$, we write $\binom{n}{r}$ instead of

$$\binom{n}{r, n-r} = \frac{n!}{r!(n-r)!}.$$

Theorem 3.4. (Multinomial Theorem and Expansion)

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{n_1 + \cdots + n_k = n \\ n_1 \geq 0, n_2 \geq 0, \ldots, n_k \geq 0}} \binom{n}{n_1, \ldots, n_k} x_1^{n_1} \cdots x_k^{n_k}.$$

Proof.

$$
\begin{aligned}
&(x_1+\cdots+x_k)^n \\
&= \underbrace{(x_1+\cdots+x_k)\cdots(x_1+\cdots+x_k)}_{n} \\
&= \sum u_1\cdots u_n \quad (u_i = x_1,\ldots,x_k,\ 1\le i\le n) \\
&= \sum \left\{ \begin{array}{c} \#\text{ of sequences of } x_1,\ldots,x_k \text{ of} \\ \text{length } n \text{ with } n_1\ x_1\text{'s}, \ldots, n_k\ x_k\text{'s} \end{array} \right\} \\
&= \sum_{\substack{n_1+\cdots+n_k=n \\ n_1\ge 0,\ldots,n_k\ge 0}} \binom{n}{n_1,\ldots,n_k} x_1^{n_1}\cdots x_k^{n_k}.
\end{aligned}
$$

□

4 Combination with Repetition

We consider combinations with repetition allowed. The number of **r-combinations of n objects with repetition allowed** is denoted by

$$\left\langle {n \atop r} \right\rangle.$$

Example 4.1. In how many ways can seven objects be taken with repetition allowed from set $A = \{a, b, c, d\}$?

Solution. Take seven objects from A with repetition allowed, say, a, a, b, b, b, c, d. This forms a multiset $\{a, a, b, b, b, c, d\}$ of seven objects. We insert 3 sticks | to separate elements of 4 different types to obtain a sequence $aa|bbb|c|d$, then convert the sequence into a sequence 0010001010 of 0 and 1 by changing letters to 0's and sticks | to 1's. For instance,

$$
\begin{array}{rcccl}
\{a,a,b,b,b,c,d\} & \leftrightarrow & aa|bbb|c|d & \leftrightarrow & 0010001010 \\
\{a,a,a,b,b,b,b\} & \leftrightarrow & aaa|bbbb|| & \leftrightarrow & 0001000011 \\
\{b,b,c,c,c,c,c\} & \leftrightarrow & |bb|ccccc| & \leftrightarrow & 1001000001 \\
\{a,c,c,c,d,d,d\} & \leftrightarrow & a||ccc|ddd & \leftrightarrow & 0110001000
\end{array}
$$

Let X be the set of 7-multisets of A, and Y the set of sequences of 0's and 1's of length 10 (=7+3) with exactly seven 0's and three 1's. The above converting actually defines a function $f : X \to Y$. Clearly, f is one-to-one and onto. Thus

$$\text{answer} = \left\langle {4 \atop 7} \right\rangle = |X| = |Y| = \binom{10}{7} = \binom{4+7-1}{7}.$$

Theorem 4.1. *The number of r-combinations of n objects with repetition allowed is*

$$\left\langle {n \atop r} \right\rangle = \binom{n+r-1}{r}.$$

Example 4.2. Eight students plan to have dinner together in a restaurant where the menu shows 20 varieties. Each student decides to order one dish and plans to share with others. How many possible combinations of eight dishes can be ordered?

$$\text{Answer } = \left\langle {20 \atop 8} \right\rangle = \binom{20+8-1}{8} = \binom{27}{8}.$$

Theorem 4.2. *The number of nonnegative integer solutions for the equation*

$$x_1 + x_2 + \cdots + x_n = r$$

is given by

$$\left\langle {n \atop r} \right\rangle = \binom{n+r-1}{r}.$$

Example 4.3. There are five types of color T-shirts on sale, black, blue, green, orange, and white. John is going to buy ten T-shirts; he has to buy at least two blues and two oranges, and at least one for all other colors. Find the number of ways that John can select ten T-shirts.

Solution. We use $1, 2, 3, 4, 5$ to denote black, blue, green, orange and white respectively. Let x_i be the number of T-shirts that John would select for the ith color T-shirt. Then the problem is to find the number of integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10, \tag{1}$$

where $x_1 \geq 1, x_2 \geq 2, x_3 \geq 1, x_4 \geq 2, x_5 \geq 1$.

Let $x_1 = y_1 + 1$, $x_2 = y_2 + 2$, $x_3 = y_3 + 1$, $x_4 = y_4 + 2$, $x_5 = y_5 + 1$. Then (1) becomes

$$y_1 + y_2 + y_3 + y_4 + y_5 = 10 - 7 = 3$$

with $y_i \geq 0$, $1 \leq i \leq 5$. Thus

$$\text{answer } = \left\langle {5 \atop 3} \right\rangle = \binom{5+3-1}{3} = \binom{7}{3} = 35.$$

Example 4.4. (a) In how many ways can a student order eight dumplings from three different kinds? (Assume that there are enough supply of dumplings of each kind.)

(b) In how many ways can a student eat five dumplings selected from the three kinds for free under the condition that dumplings of the same kind should be eaten consecutively one by one?

(c) The same question as (b) having 5 changed to n and 3 changed to m.

Solution. (a) $\left\langle {3 \atop 8} \right\rangle = \binom{10}{8} = 45$.

(b) There are $\binom{3}{k}$ ways to select k types of dumplings, $1 \leq k \leq 3$.

For each selected k types of dumplings, to select 5 dumplings so that each kind is selected, it can be done as follows: Select one from each kind first. Then select $5 - k$ dumplings from the k types with repetition allowed. Thus there are $\left\langle {k \atop 5-k} \right\rangle$ such selections.

For each selected k types of dumplings and a selection of 5 dumplings from the k types so that each kind is selected, there are exactly $k!$ ways to eat the dumplings. Thus

$$\text{answer} = \sum_{k=1}^{3} k! \binom{3}{k} \left\langle {k \atop 5-k} \right\rangle = 63.$$

(c) answer $= \sum_{k=1}^{m} k! \binom{m}{k} \left\langle {k \atop n-k} \right\rangle$.

5 Combinatorial Proof

Using bijection to prove an identity is sometimes called a **combinatorial proof**. Here are a few examples.

Example 5.1. The identity

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

gives the **Yang Hui Triangle** (or **Pascal Triangle**).

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Proof. Let $a_1, a_2, \ldots, a_{n+1}$ be distinct elements. Consider the sets

$$\begin{aligned}
A_n &= \{a_1, a_2, \ldots, a_n\}, \\
A_{n+1} &= \{a_1, a_2, \ldots, a_n, a_{n+1}\}.
\end{aligned}$$

The r-subsets of A_{n+1} are divided into two types.

Type I: r-subsets of A_n;
Type II: r-subsets of A_{n+1}, but not subsets of A_n.

There are $\binom{n}{r}$ r-subsets of Type I. Each r-subset of Type II must contain the element a_{n+1}; and each such r-subset can be obtained by taking an $(r-1)$-subsets of A_n first then adding the element a_{n+1} to it. Thus there are $\binom{n}{r-1}$ r-subsets of Type II. Adding the number of r-subsets of two types, we have $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$. □

Example 5.2.

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

Proof. Consider n-combinations of $2n$ distinct balls, of which n balls are white and the other n balls are black. Each n-combination can be obtained by taking k balls from the n white balls and $(n-k)$ balls from the n black balls, where

$0 \le k \le n$. We then have

$$\begin{aligned}\binom{2n}{n} &= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} \\ &= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.\end{aligned}$$

In general,

$$\binom{m+n}{k} = \sum_{\substack{i+j=k \\ i,j\ge 0}} \binom{m}{i}\binom{n}{j}.$$

$\square$

6 Relation to Probability

There are many problems about chances or possibilities, called **probability** in mathematics. When we flip a HK dollar coin, we have two possible **outcomes**, Number and Flower. If the coin is fair, the chance to have the outcome, Flower, is 50%. When we roll a pair of dice we may have outcomes, a collection of pairs of numbers from 1 to 6. The chance of the event of the outcomes that the sum of the pair is even is one-half. For instance, we may be interested in finding the probability of the event of the outcomes that the sum of the pair is 8.

Definition 6.1. A collection of outcomes in a probabilistic experiment is called an **event**. If each outcome is equally likely to be happened, we define

$$\begin{aligned}P(A) &= \textbf{Probability of an event } A \\ &= \frac{\text{number of favorite outcomes}}{\text{number of possible outcomes}}. \qquad (2)\end{aligned}$$

Example 6.1. What is the probability of selecting seven distinct numbers from $1, 2, \ldots, 11$ so that two are less than 5, one is equal to 5, and four are larger than 5?

Solution. The number of possible outcomes equals $\binom{11}{7}$; the number of favorite outcomes equals $\binom{4}{2}\binom{1}{1}\binom{6}{4}$. Thus

$$\text{probability} = \frac{\binom{4}{2}\binom{1}{1}\binom{6}{4}}{\binom{11}{7}}.$$

Example 6.2. Find the probability that no two persons have the same birthday in a party of 40 people.

Solution. The number of possible outcomes is 365^{40}. The number of favorite outcomes is $\binom{365}{40}40!$. Then

$$\text{probability} = \frac{\binom{365}{40}40!}{365^{40}} \approx 0.109.$$

Example 6.3. What is the probability of rolling a pair of dice so that the sum of numbers on the top facets equals 8?

First Method: Since there is no order between the two dice, there are 21 possible outcomes

$$\{i, j\}, \ 1 \le i \le j \le 6$$

and three favorite outcomes $\{2,6\}$, $\{3,5\}$, $\{4,4\}$. So the the probability is $\frac{3}{21} = \frac{1}{7} \approx 0.14286$.

Second Method: One may color the two dice as black and white so that the two dice are ordered. There are $36(= 6 \times 6)$ possible outcomes and five favorite outcomes

$$(2,6), \ (3,5), \ (4,4), \ (5,3), \ (6,2).$$

The probability equals $\frac{5}{36} \approx 0.13889$.

Which method is correct and why?

Example 6.4. Find the probability of rolling four dice simultaneously so that the sum of points equals 9.

Solution. The number of possible outcomes is 6^4. The number of favorite outcomes is the number of positive integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 9,$$

which is equal to the number of nonnegative integer solutions of the equation

$$y_1 + y_2 + y_3 + y_4 = 5.$$

Thus

$$\text{probability} = \frac{\left\langle {4 \atop 5} \right\rangle}{6^4} = \frac{7}{162} \approx \frac{1}{23}.$$

7 Inclusion-Exclusion Principle

Let U be a finite set. For two subsets $A_1, A_2 \subseteq U$, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Equivalently,

$$|\bar{A}_1 \cap \bar{A}_2| = |U| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

For three subsets $A_1, A_2, A_3 \subseteq U$, we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad -|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ &\quad +|A_1 \cap A_2 \cap A_3|, \end{aligned}$$

Equivalently,

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= |U| - |A_1| - |A_2| - |A_3| \\ &\quad +|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad -|A_1 \cap A_2 \cap A_3|. \end{aligned}$$

These formulas and similar kinds for more subsets are called the **Inclusion-Exclusion Principle**.

Example 7.1. The pin numbers of Hang Seng Bank card are 6-digit nonnegative integers. How many possible pin numbers can be made so that a triple 444 doesn't appear?

Solution. Let U be the set of all possible pin numbers. Then $|U| = 10^6$. Let

$$\begin{aligned} A_1 &= \text{set of pin numbers of the form } 444xxx, \\ A_2 &= \text{set of pin numbers of the form } x444xx, \\ A_3 &= \text{set of pin numbers of the form } xx444x, \\ A_4 &= \text{set of pin numbers of the form } xxx444, \end{aligned}$$

where x varies from 0 to 9. Then

$$|A_1| = |A_2| = |A_3| = |A_4| = 10^3;$$

$$\begin{gathered} |A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_4| = 10^2, \\ |A_1 \cap A_3| = |A_2 \cap A_4| = 10, \\ |A_1 \cap A_4| = 1; \end{gathered}$$

$$|A_1 \cap A_2 \cap A_3| = |A_2 \cap A_3 \cap A_4| = 10,$$

$$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = 1;$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 1.$$

Thus the number of pin numbers equals

$$10^6 - 4 \cdot 10^3 + (3 \cdot 10^2 + 2 \cdot 10 + 1) - (2 \cdot 10 + 2) + 1 = 996310.$$

Example 7.2. Find the number of positive integer solutions for the linear equation $x_1 + x_2 + x_3 = 8$.

Solution. Let U be the set of all nonnegative integer solutions of the equation $x_1 + x_2 + x_3 = 8$. Let $A_i \subset U$ be subsets such that $x_i = 0$, $1 \leq i \leq 3$. Then

$$|U| = \left\langle {3 \atop 8} \right\rangle = \binom{3+8-1}{8} = \binom{10}{8} = 45;$$

$$|A_1| = |A_2| = |A_3| = 9;$$

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = 1;$$

$$|A_1 \cap A_2 \cap A_3| = 0.$$

By the Inclusion-Exclusion Principle,

$$\begin{aligned}|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= |U| - |A_1| - |A_2| - |A_3| \\ &\quad +|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad -|A_1 \cap A_2 \cap A_3| \\ &= 45 - 3 \cdot 9 + 3 \cdot 1 - 0 = 21.\end{aligned}$$

The fastest way to get the answer is to apply substitution $x_i = y_i + 1$, $1 \le i \le 3$. Then the answer is the number of nonnegative integer solutions of $y_1 + y_2 + y_3 = 5$, namely,

$$\left\langle {3 \atop 5} \right\rangle = \binom{3+5-1}{5} = \binom{7}{5} = 21.$$

Theorem 7.1. *Let U be a finite set and $A_i \subset U$ be subsets, $1 \le i \le n$. Then*

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = {} & |A_1| + |A_2| + \cdots |A_n| \\ & - \big(|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_1 \cap A_n| \\ & \quad +|A_2 \cap A_3| + |A_2 \cap A_4| + \cdots + |A_2 \cap A_n| \\ & \quad + \cdots + |A_{n-1} \cap A_n|\big) \\ & + \big(|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|\big) \\ & - \cdots \\ & +(-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n| \\ = {} & \textstyle\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left|A_{i_1} \cap \cdots \cap A_{i_k}\right|.\end{aligned} \tag{3}$$

Equivalently,

$$\begin{aligned}|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| = {} & |U| - \textstyle\sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| \\ & - \textstyle\sum_{i<j<k} |A_i \cap A_j \cap A_k| \\ & + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n| \\ = {} & |U| + \textstyle\sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} \left|A_{i_1} \cap \cdots \cap A_{i_k}\right|.\end{aligned} \tag{4}$$

Proof. For each element $x \in U$, we show that x contributes the same count to both sides of (4).

Case I: $x \notin A_1 \cup A_2 \cup \cdots \cup A_n$.

The element x is counted once on the left of (4) in

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n;$$

and is counted once in U and 0 times in all subsets

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}, \quad \text{where} \quad i_1 < i_2 < \cdots < i_k.$$

Thus x is counted once on both sides left and right sides of (4).

Case II: $x \in A_1 \cup A_2 \cup \cdots \cup A_n$.

We assume that x belongs to exactly r subsets of

$$A_1, A_2, \ldots, A_n, \quad \text{say,} \quad A_{t_1}, A_{t_2}, \ldots, A_{t_r}.$$

Then x is counted $\binom{r}{0}$, $\binom{r}{1}$, $\binom{r}{2}$, $\binom{r}{3}$, ..., $\binom{r}{r}$ times in

$$U, \quad \sum_i |A_{t_i}|, \quad \sum_{i<j} |A_{t_i} \cap A_{t_j}|, \quad \sum_{i<j<k} |A_{t_i} \cap A_{t_j} \cap A_{t_k}|,$$

$$\ldots, \quad |A_{t_1} \cap A_{t_2} \cap \cdots \cap A_{t_r}|, \quad \text{respectively.}$$

Consequently, the contribution of x on the right side of (4) is

$$\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r}$$
$$= \Big(1 + (-1)\Big)^r = 0.$$

Therefore x is counted zero times on both sides of (4). $\square$

Remark. The subsets $A_i \subset U$ are usually given by conditions (or properties) c_i on the elements of U, $1 \le i \le n$. Let

$$\begin{aligned}
N &= |U|; \\
\bar{N} &= \#\text{ of objects satisfying none} \\
&\quad\ \text{of the conditions } c_1, \ldots, c_n; \\
N(c_{i_1} c_{i_2} \cdots c_{i_k}) &= \#\text{ of objects satisfying} \\
&\quad\ \text{conditions } c_{i_1}, c_{i_2}, \cdots, c_{i_k}, \\
&\quad\ 1 \le i_1 < i_2 < \cdots < i_k \le n.
\end{aligned}$$

Then the Inclusion-Exclusion Principle can be stated as

$$\begin{aligned}
\bar{N} \;=\; N \;-\; & \Big(N(c_1)+N(c_2)+\cdots+N(c_n)\Big) \\
+\; & \Big(N(c_1c_2)+N(c_1c_3)+\cdots+N(c_1c_n) \\
& \quad +N(c_2c_3)+N(c_2c_4)+\cdots+N(c_2c_n) \\
& \qquad\qquad +\cdots+N(c_{n-1}c_n)\Big) \\
+\; & \Big(N(c_1c_2c_3)+\cdots+N(c_{n-2}c_{n-1}c_n)\Big) \\
+\; & \cdots+(-1)^n N(c_1c_2\cdots c_n). \qquad (5)
\end{aligned}$$

Let

$$N_k \;=\; \sum_{i_1<i_2<\cdots<i_k} |A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}|.$$

Then the Inclusion-Exclusion Formula can be stated as

$$|\bar{A}_1\cap\bar{A}_2\cap\cdots\cap\bar{A}_n| = N_0-N_1+N_2-N_3+\cdots+(-1)^nN_n. \qquad (6)$$

Example 7.3. Find the number of possible Hong Kong telephone numbers having no 14?

Solution. Let U be the set of Hong Kong telephone numbers. Obviously, $|U|=10^8$. Let

$$\begin{aligned}
A_i \;=\;& \text{set of HK phone numbers whose } i\text{th digit} \\
& \text{is 1 and } (i+1)\text{th digit is 4}, \quad 1\le i\le 7.
\end{aligned}$$

Then

$$\begin{aligned}
N_1 &= \sum_{i=1}^{7}|A_i| = \binom{7}{1}10^6, \\
N_2 &= \sum_{1\le i<j\le 7}|A_i\cap A_j| = \binom{6}{2}10^4, \\
N_3 &= \sum_{1\le i<j<k\le 7}|A_i\cap A_j\cap A_k| = \binom{5}{3}10^2, \\
N_4 &= \sum_{1\le i<j<k<l\le 7}|A_i\cap A_j\cap A_k\cap A_l| = \binom{4}{4}10^0.
\end{aligned}$$

Thus the answer is

$$10^8 - \binom{7}{1} 10^6 + \binom{6}{2} 10^4 - \binom{5}{3} 10^2 + \binom{4}{4} 10^0 = 93149001.$$

8 Algebraic Proof of Inclusion-Exclusion Formula

Given a finite universal set U. Recall the **characteristic function** of subset A of U is a function $1_A : U \to \mathbb{R}$ defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For functions $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$, we define their **addition** and **subtraction** as the following functions

$$\begin{array}{ll} f + g : U \to \mathbb{R}, & (f+g)(x) = f(x) + g(x), \\ f - g : U \to \mathbb{R}, & (f-g)(x) = f(x) - g(x); \end{array}$$

their **multiplication** as

$$fg : U \to \mathbb{R}, \quad fg(x) = f(x)g(x);$$

and the **scalar multiplication** for a constant real number a as

$$af : U \to \mathbb{R}, \quad (af)(x) = af(x).$$

The set of functions on a given set together with the addition and scalar multiplication forms a vector space, called the **vector space of functions**. Mover, the product of functions make the vector space of function into an algebra. Note that the following so-called **distributive law**:

$$f(g+h) = fg + fh.$$

Proposition 8.1. *Let $A, B \subseteq U$ and $f : U \to \mathbb{R}$ be a function. Then*

(a) $1_{A\cap B} = 1_A 1_B$,

(b) $1_{\bar{A}} = 1_U - 1_A$, *where* $\bar{A} = U - A$,

(c) $1_U f = f$,

(d) $1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}$.

For each function $f : U \to \mathbb{R}$, we define its **weight** as the number

$$w(f) = \sum_{x\in U} f(x).$$

Then for real numbers $a, b \in \mathbb{R}$,

$$w(af + bg) = aw(f) + bw(g).$$

Note that $\bar{A}_1 \cap \cdots \cap \bar{A}_n$ is the set of elements of U satisfying none of the conditions $c_1, \ldots, c_n$. The set $A_{i_1} \cap \cdots \cap A_{i_k}$ consists of the elements of U satisfying the conditions $c_{i_1}, \ldots, c_{i_k}$. On the one hand by Proposition 8.1, we have

$$\begin{aligned}
1_{\bar{A}_1\cap\cdots\cap\bar{A}_n} &= 1_{\bar{A}_1}\cdots 1_{\bar{A}_n} \\
&= (1_U - 1_{A_1})\cdots(1_U - 1_{A_n}) \\
&= \sum f_1 f_2 \cdots f_n \text{ (each } f_i \text{ is either } 1_U \text{ or } -1_{A_i}) \\
&= \underbrace{1_U\cdots 1_U}_{n} + \sum_{\substack{1\le i_1<\cdots<i_k\le n \\ 1\le k\le n}} \underbrace{1_U\cdots 1_U}_{n-k}(-1_{A_{i_1}})\cdots(-1_{A_{i_k}}) \\
&= 1_U + \sum_{k=1}^{n}(-1)^k \sum_{1\le i_1<\cdots<i_k\le n} 1_{A_{i_1}}\cdots 1_{A_{i_k}} \\
&= 1_U + \sum_{k=1}^{n}(-1)^k \sum_{1\le i_1<\cdots<i_k\le n} 1_{A_{i_1}\cap\cdots\cap A_{i_k}}.
\end{aligned}$$

On the other hand,

$$1_{\bar{A}_1\cap\cdots\cap\bar{A}_n} = 1_{\overline{A_1\cup\cdots\cup A_n}} = 1_U - 1_{A_1\cup\cdots\cup A_n}.$$

Then

$$1_{A_1\cup\cdots\cup A_n} = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le n} 1_{A_{i_1}\cap\cdots\cap A_{i_k}}.$$

Thus

$$\begin{aligned}
|A_1 \cup \cdots \cup A_n| &= w(1_{A_1 \cup \cdots \cup A_n}) \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} w\left(1_{A_{i_1} \cap \cdots \cap A_{i_k}}\right) \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|.
\end{aligned}$$

9 Generalized Inclusion-Exclusion Formula

Theorem 9.1. *Let U be a finite set, $|U| = N$. Let $c_1, \ldots, c_n$ be some properties about elements of U. Let*

$$E_m = \{x \in U : x \text{ sarisfies exactly } m \text{ properties of } c_1, \ldots, c_n.\}$$

Then

$$\begin{aligned}
|E_m| \; = \; & \binom{m}{0} N_m - \binom{m+1}{1} N_{m+1} + \binom{m+2}{2} N_{m+2} \\
& - \quad \cdots \quad + \quad (-1)^{n-m} \binom{n}{n-m} N_n.
\end{aligned} \tag{7}$$

Proof. It suffices to prove the following identity:

$$1_{E_m} = \sum_{j=0}^{n-m} (-1)^j \binom{m+j}{j} \sum_{i_1 < \cdots < i_{m+j}} 1_{A_{i_1} \cap \cdots \cap A_{i_{m+j}}}. \tag{8}$$

The identity (7) follows from the (8) by applying the operator w to both sides.

For each $x \in U$, we count the contribution of x on both sides of (8). We divide the situations into three cases.

Case I: x satisfies fewer than m conditions.

In this case the contributions of x on both sides are 0.

Case II: x satisfies exactly m conditions, say, $c_{i_1}, \cdots, c_{i_m}$.

The contribution of x on the left side is 1. The contribution of x on the right side is also 1 because x is counted once in N_m and 0 times in all N_k for $k > m$.

Case III: x satisfies exactly r conditions, say, $c_{i_1}, \cdots, c_{i_r}$, and $r > m$.

The contribution of x to the left side is 0. On the right side, the contributions of x to N_m, N_{m+1}, ..., N_r are

$$\binom{r}{m}, \quad \binom{r}{m+1}, \quad \ldots, \quad \binom{r}{r}, \quad 0, \quad \ldots, \quad 0$$

respectively; and the contributions of x to N_k with $k > r$ are all 0. Thus the contribution of x to the right side is

$$\binom{m}{0}\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}.$$

Now it is easy to see that

$$\begin{aligned}
&\sum_{j=0}^{r-m}(-1)^j\binom{m+j}{j}\binom{r}{m+j} \\
&\quad = \sum_{j=0}^{r-m}(-1)^j\frac{(m+j)!}{j!m!}\cdot\frac{r!}{(m+j)!(r-m-j)!} \\
&\quad = \sum_{j=0}^{r-m}(-1)^j\frac{r!}{m!(r-m)!}\cdot\frac{(r-m)!}{i!(r-m-j)!} \\
&\quad = \sum_{i=0}^{r-m}(-1)^j\binom{r}{m}\binom{r-m}{j} \\
&\quad = \binom{r}{m}\sum_{j=0}^{r-m}(-1)^j\binom{r-m}{j} \\
&\quad = \binom{r}{m}\big[(-1)+1\big]^{r-m} = 0.
\end{aligned}$$

Therefore the contributions of x on both sides are the same. □

10 Pigeonhole Principle

Theorem 10.1. (Pigeonhole Principle) *If n objects are placed into m boxes, $n > m$, then there is at least one box containing more than one object.*

The Pigeonhole Principle is a common Chinese saying: When pigeons are put in pigeonholes, if pigeons are more than pigeonholes, there must be at least two pigeons put in a same pigeonhole.

Example 10.1. Among any five integers between 1 and 8 inclusive, there are at least two of them adding up to 9.

Solution. We can divide the set $\{1, 2, \ldots, 8\}$ into four disjoint subsets where each has two elements adding up to 9:

$$\{1, 8\}, \quad \{2, 7\}, \quad \{3, 6\}, \quad \{4, 5\}.$$

When selecting five numbers from these four subsets, at least two of the five selected numbers must come from a same subset of the four subsets. Thus their addition is 9.

Example 10.2. Show that in any group of two or more persons there are at least two having the same number of friends. (It is assumed that if a person x is a friend of a person y then y is also a friend of x).

Solution. Consider a group of n persons, $n \geq 2$. The number of friends of a person x should be an integer k, where $0 \leq k \leq n-1$. If there is one person x^* whose number of friends is $n-1$, then everyone is a friend of x^*. Thus 0 and $n-1$ can not be simultaneously the numbers of friends of some people in the group. The Pigeonhole Principle tells us that there are at least two people having the same number of friends.

Example 10.3. Show that if $a_1, a_2, \ldots, a_k$ are integers (not necessarily distinct), then some of them can be added up to a multiple of k.

Solution. Consider the following $k+1$ integers (not necessarily distinct):

$$0,\ a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \ldots,\ a_1 + a_2 + \cdots + a_k. \tag{9}$$

By the Division Algorithm, the remainders of any integer dividing by k can only be $0, 1, 2, \ldots, k-1$. By the Pigeonhole Principle, there are at least two integers in (9), say,

$$a_1 + \cdots + a_i \quad \text{and} \quad a_1 + \cdots + a_j$$

having the same remainder dividing by k. (The integer $a_1 + \cdots + a_i$ could be 0, the very first integer in (9).) Thus the difference

$$(a_1 + \cdots + a_j) - (a_1 + \cdots + a_i) = a_{i+1} + a_{i+2} + \cdots + a_j$$

is a multiple of k.

Example 10.4. Given 10 distinct integers $a_1, a_2, \ldots, a_{10}$ such that $0 \le a_i < 100$. Is there any subset of $\{a_1, \ldots, a_{10}\}$ such that the sum of numbers of the subset with sign is zero?

Solution. Consider all possible partial sums of the selected numbers $a_1, a_2, \ldots, a_{10}$. The values of these sums should be between 0 and 1000. Note that the number of subsets of 10 objects is $2^{10} = 1024$. By the Pigeonhole Principle, there are at least two subsets $A, B \subset \{a_1, a_2, \ldots, a_{10}\}$ such that

$$\sum_{a_i \in A} a_i = \sum_{a_j \in B} a_j.$$

Move the numbers on the right side to the left; the numbers of the intersection $A \cap B$ will be canceled. Thus, the sum of integers of $A \Delta B$ with positive sign in $A - B$ and negative sign in $B - A$ equals 0.

Theorem 10.2. *If n objects are placed in m boxes, then one of the boxes must contain at least $\lceil \frac{n}{m} \rceil$ objects.*

Problem Set 3

1. A computer user name consists of three English letters followed by five digits. How many different user names can be made?

   Answer: $26^3 \cdot 10^5$.

2. A set lunch includes a soup, a main course, and a drink. Suppose a customer can select from three kinds of soup, five main courses, and four kinds of drink. How many varieties of set lunches can be possibly made?

   Answer: $3 \cdot 5 \cdot 4$.

3. (Not required) Find a procedure to determine the number of zeros at the end of the integer $n!$ written in base 10. Justify your procedure and make examples for 12! and 26!.

4. How many different words can be made by rearranging the order of letters in the word HONGKONG?

   Answer: $\binom{8}{2,2,2,1,1}$

5. A bookshelf is to be used to exhibit ten math books. There are eight kinds of books on *Calculus*, six kinds of books on *Linear Algebra*, and five kinds of books on *Discrete Mathematics*. Books of the same subject should be displayed together.

   (a) In how many ways can ten distinct books be exhibited so that there are five *Calculus* books, three *Linear Algebra* books, and two *Discrete Mathematics* books?

   (b) In how many ways can ten books (not necessarily distinct) be exhibited so that there are five *Calculus* books, three *Linear Algebra* books, and two *Discrete Mathematics* books?

   Solution: (a) $\binom{8}{5}\binom{6}{3}\binom{5}{2} \cdot 5!3!2! \cdot 3!$.

   (b) $8^5 \cdot 6^3 \cdot 5^2 \cdot 3!$.

6. There are n men and n women to form a circle (line), $n \geq 2$. Assume that all n men are indistinguishable and all n women are indistinguishable.

   (a) How many possible patterns of circles (lines) could be formed so that men and women alternate?

   (b) How many possible patterns of lines can be formed so that each man is next to at least one woman?

   (c) How many possible patterns of circles can be formed so that each man is next to at least one woman and each woman is next to at least one man?

   SOLUTION. Case I: All n men are indistinguishable and all n women are indistinguishable.

(a) Circle case: 1. Line case: 2.

(b) It is clear that no three or more men can be seated together.

Line Case. Let k denote the number of pairs of men in a seating plan, where $0 \le k \le \lfloor \frac{n}{2} \rfloor$. We may think of two men seated next to each other as one big man. Then there are k big men and $n-2k$ men. So there are total $n-k$ Men (men and big men). The number of seating plans for these $n-k$ Men on a line is the number words of zeros and ones with exactly k ones and exactly $n-2k$ zeros, which is $\binom{n-k}{k}$. Putting $n-k-1$ women between these $n-k$ Men, there are $k+1$ remaining women. These $k+1$ women can be arbitrarily distributed into $n-k+1$ locations among the $n-k$ Men; there are $\left\langle {n-k+1 \atop k+1} \right\rangle = \binom{n+1}{k+1}$ ways to distribute the $k+1$ women. Thus the total number of required seating plans is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \binom{n+1}{k+1}.$$

Circle case: Let k be the number of pairs of man seating together. Then $0 \le k \le \lfloor \frac{n}{2} \rfloor$, and there are also k pairs of women seating together. There are $n-2k$ single man and k double-man; there are also $n-2k$ single women and k double women, seating alternately. Then the number of seating patterns with k double-men and k double-women is

$$\frac{1}{n-k} \sum_{d \mid \gcd(k,n)} \binom{(n-k)/d}{k/d} \phi(d),$$

where $\phi(d)$ is the number of integers in $[1, d]$ coprime to d. The total number of seating patterns is

$$2 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{n-k} \sum_{d \mid \gcd(k,n)} \binom{(n-k)/d}{k/d} \phi(d).$$

7. Four identical six-sided dice are tossed simultaneously and numbers showing on the top faces are recorded as a multiset of four elements. How many different multisets are possible?

Answer: $\left\langle {6 \atop 4} \right\rangle$.

8. Find the number of non-decreasing coordinate paths from the origin $(0,0,0)$ to the lattice point (a,b,c).

   Answer: $\binom{a+b+c}{a,b,c}$.

9. How many six-card hands can be dealt from a deck of 52 cards?

   Answer: $P_6^{52} = \binom{52}{6}6!$ (a hand is ordered); $\binom{52}{6}$ (a hand is unordered).

10. How many different eight-card hands with five red cards and three black cards can be dealt from a deck of 52 cards?

    Answer: $\binom{26}{5}\binom{26}{3}8!$ (ordered); $\binom{26}{5}\binom{26}{3}$ (unordered).

11. Fortune draws are arranged to select six ping pang balls simultaneously from a box in which 20 are orange and 30 are white. A draw is lucky if it consists of three orange and three white balls. What is the chance of a lucky draw?

    Answer: $\frac{\binom{20}{3}\binom{30}{3}}{\binom{50}{6}}$.

12. Determine the number of integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 \le 38,$$

where

(a) $x_i \ge 0$, $1 \le i \le 4$.

(b) $x_1 \ge 0$, $x_2 \ge 2$, $x_3 \ge -2$, $3 \le x_4 \le 8$.

Solution: (a) Note that there is a one-to-one correspondence between the set of solutions for the inequality

$$x_1 + x_2 + x_3 + x_4 \le 38$$

and the set of solutions for the equality

$$x_1 + x_2 + x_3 + x_4 + x_5 = 38$$

with $x_5 \ge 0$. Thus the number of solutions for the inequality is

$$\left\langle {5 \atop 38} \right\rangle = \binom{5+38-1}{38}.$$

(b) Let $x_1 = y_1$, $x_2 = y_2 + 2$, $x_3 = y_3 - 2$, $x_4 = y_4 + 3$. Then the equation becomes $y_1 + y_2 + y_3 + y_4 \le 35$ with $y_i \ge 0$ for $1 \le i \le 4$ and $y_4 \le 5$.

Solution: The number of solutions is the same as the number of solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 35$ with $y_4 \le 5$.

Answer: $\sum_{i=0}^{5} \left\langle {4 \atop 35-i} \right\rangle = \sum_{i=0}^{5} \binom{38-i}{35-i}$

13. Determine the number of nonnegative integer solutions to the pair of equations

$$x_1 + x_2 + x_3 = 8, \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18.$$

Solution: We need to count the number of tuples $(x_1, \ldots, x_6)$ satisfying both equations. Answer: $\left\langle {3 \atop 8} \right\rangle \left\langle {3 \atop 10} \right\rangle = \binom{10}{8}\binom{12}{10}$.

14. Show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have the same sum.

Solution: (a) The six selected numbers are not necessarily distinct. Note that

$$\underbrace{1 + \cdots + 1}_{6} = 6, \quad \underbrace{15 + \cdots + 15}_{6} = 90.$$

There are 85 integers in $[6, 90]$. However, $\left\langle {15 \atop 6} \right\rangle = \binom{20}{6} = 38760$. Then are at least $38760/85 = 456$ ways to choose six integers (not necessarily distinct) whose sums are the same. (b) The six selected numbers are distinct. Note that $1+2+\cdots+6 = 21$ and $10+11+\cdots+15 = 75$. There are 55 integers in $[21, 75]$. However, there are $\binom{15}{6} = 5005$ ways to select six integers. Thus on average there are $5005/55 = 91$ selections have the same sum.

15. Show that if five points are selected in a square whose sides have length 2, then there are at leat two points whose distance is at most $\sqrt{2}$.

Answer: Dividing the square of length 2 into 4 squares of length 1. Then at least two points must be in one small square, and the distance is less than or equal to $\sqrt{2}$.

16. Prove that if any 14 numbers (not necessarily distinct) from 1 to 25 are chosen, then one of them is a multiple of another.

Solution: Dividing integers in $[1, 25]$ into 13 pigeonholes

$$\{1,2,4,8,16\}, \{3,6,12,24\}, \{5,10,20\}, \{7,14\}, \{9,18\},$$

$$\{11,22\}, \{13\}, \{15\}, \{17\}, \{19\}, \{21\}, \{23\}, \{25\}.$$

17. Twenty disks labelled 1 through 20 are placed face down on a table. Disks are selected (by a player) one at a time and turned over until 10 disks have been chosen. If the labels of two disks add up to 21, the player loses. Is it possible to win this game?

    Answer: Yes. Dividing 20 integers $1, 2, \ldots, 20$ into pigeonholes

    $$\{1,20\}, \{2,19\}, \{3,18\}, \{4,17\}, \{5,16\},$$

    $$\{6,15\}, \{7,14\}, \{8,13\}, \{9,12\}, \{10,11\}.$$

    The chance the player can win the game is $2^{10}/\binom{20}{10}$.

18. Show that it is impossible to arrange the numbers $1, 2, \ldots, 10$ in a circle so that every triple of consecutively placed numbers has a sum less than 15.

    Solution: Let the 10 numbers be arranged as $a_1, a_2, \ldots, a_{10}$ in a circle so a_1 is next to a_{10}. Then $a_1+a_2+a_3 \le 14$, $a_4+a_5+a_6 \le 14$, $a_7+a_8+a_9 \le 14$, and $a_{10} \le 10$. Thus $a_1 + a_2 + \cdots + a_{10} \le 3 \cdot 14 + 10 = 52$. However, $a_1 + a_2 + \cdots + a_{10} = 1 + 2 + \cdots + 10 = 55$, a contradiction.

19. In how many ways to arrange the letters E, I, M, O, T, U, Y so that YOU, ME and IT would not occur?

    Solution: Let U be the set of all arrangements of E, I, M, O, T, U, Y. Let A, B, and C be the sets of arrangements with YOU, ME, and IT respectively. Then

    $$\begin{aligned} |\bar{A} \cap \bar{B} \cap \bar{C}| &= |U| - |A| - |B| - |C| + \\ &\quad |A \cap B| + |A \cap C| + |B \cap C| - \\ &\quad |A \cap B \cap C| \\ &= 7! - 5! - 6! - 6! + 4! + 4! + 5! - 3!. \end{aligned}$$

20. Six passengers have a trip by taking a van of six seats. Passengers randomly select their seats. When the van stops for a break, every passenger left the van.

    (a) What is the chance that the seat of every passenger after a break is the same as their seat before the break?

    (b) What is the chance that exactly five passengers have the same seats before and after a break?

    (c) What is the probability that at least one passenger has the same seat before and after a break?

    Solution: (a) $1/6!$; (b) 0; (c) We find the probability p that no one got his/her original seat. Let U be the set of all possible seating plans. Let A_i be the set of all possible seating plans that the ith person got his original seat, $1 \leq i \leq 6$. Then

$$\begin{aligned} p &= |U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots \\ &= 6! - \binom{6}{1} 5! + \binom{6}{2} 4! - \binom{6}{3} 3! + \\ &\quad \binom{6}{4} 2! - \binom{6}{5} 1! + \binom{6}{6}. \end{aligned}$$

21. (Note required) Let M be a multiset of type $(n_1, n_2, \ldots, n_k)$ such that $n_i \geq 1$ for $1 \leq i \leq k$. If the numbers $n_1, n_2, \ldots, n_k$ are all coprime with $n = n_1 + n_2 + \cdots + n_k$, then the number of round permutations of M is

$$\frac{\binom{n}{n_1, n_2, \ldots, n_k}}{n}.$$

    The formula is actually valid when $\gcd(n_1, \ldots, n_k) = 1$, but we didn't define the gcd yet for more than two integers. Find a counterexample if the conditions are not satisfied.

22. (Not required) Find the number of nondecreasing lattice paths from the origin $(0, 0)$ to a non-negative lattice point (a, b), allowing only horizontal,

vertical, and diagonal unit moves; that is, allowing moves

$$\begin{aligned}
(x,y) &\to (x+1,y),\\
(x,y) &\to (x,y+1),\\
(x,y) &\to (x+1,y+1).
\end{aligned}$$

Hint: For any such path with k diagonal moves ($0 \le k \le \min\{a,b\}$), the number of horizontal moves should be $a-k$ and the number of vertical moves should be $b-k$. Thus

$$\text{answer: } \sum_{k=0}^{\min\{a,b\}} \binom{a+b-k}{a-k,\, b-k,\, k}.$$

23. (Not required) **Thinking problem.** Find the number of nondecreasing lattice paths from the origin $(0,0)$ to a nonnegative lattice point (a,b), allowing arbitrary straight moves from one lattice point to another lattice point so that no lattice points on the line between two lattice points; that is, allowing all moves

$$(x,y) \to (x+k, y+h),$$

where $k,h \in \mathbb{N}$, $(h,k) \neq (0,0)$, $\gcd(k,h)=1$.
(Answer: unknown)