Math2343: Problem Set 4

(Deadline: 11 Nov. 2013)

1. Let R be a binary relation from X to Y, $A, B \subseteq X$.

   (a) If $A \subseteq B$, then $R(A) \subseteq R(B)$.

   (b) $R(A \cup B) = R(A) \cup R(B)$.

   (c) $R(A \cap B) \subseteq R(A) \cap R(B)$.

   Proof. (a) For any $y \in R(A)$, there is an $a \in A$ such that $(a, y) \in R$. Obviously, $a \in B$. Thus $b \in R(B)$.

   (b) Since $R(A) \subseteq R(A \cup B)$, $R(B) \subseteq R(A \cup B)$, we have $R(A) \cup R(B) \subseteq R(A \cup B)$. On the other hand, for any $y \in R(A \cup B)$, there is an $x \in A \cup B$ such that $(x, y) \in R$. Then either $x \in A$ or $x \in B$. Thus $y \in R(A)$ or $y \in R(B)$; i.e., $y \in R(A) \cap R(B)$. Therefore $R(A) \cup R(B) \supseteq R(A \cup B)$.

   (c) It follows from (a) that

   $$R(A \cap B) \subseteq R(A) \quad \text{and} \quad R(A \cap B) \subseteq R(B).$$

   Hence $R(A \cap B) \subseteq R(A \cap B)$. □

2. Let R_1 and R_2 be relations from X to Y. If $R_1(x) = R_2(x)$ for all $x \in X$, then $R_1 = R_2$.

   Proof. For any $(x, y) \in R_1$, we have $y \in R_1(x)$. Since $R_1(x) = R_2(x)$, then $y \in R_2(x)$. Thus $(x, y) \in R_2$. Similarly, for any $(x, y) \in R_2$, we have $(x, y) \in R_2$. Hence $R_1 = R_2$. □

3. Let $a, b, c \in \mathbb{R}$. Then

   $$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
   $$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

   Proof. Note that the cases $b < c$ and $b > c$ are equivalent. There are three essential cases to be verified.

   Case 1: $a < b < c$. We have

   $$a \wedge (b \vee c) = a = (a \wedge b) \vee (a \wedge c),$$
   $$a \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c).$$

   Case 2: $b < a < c$. We have

   $$a \wedge (b \vee c) = a = (a \wedge b) \vee (a \wedge c),$$
   $$a \vee (b \wedge c) = a = (a \vee b) \wedge (a \vee c).$$

   Case 3: $b < c < a$. We have

   $$a \wedge (b \vee c) = c = (a \wedge b) \vee (a \wedge c),$$
   $$a \vee (b \wedge c) = a = (a \vee b) \wedge (a \vee c).$$

   □

4. Let $R_i \subseteq X \times Y$ be a family of relations from X to Y, $i \in I$.

   (a) If $R \subseteq W \times X$, then $R\left(\bigcup_{i\in I} R_i\right) = \bigcup_{i\in I} RR_i$;

   (b) If $S \subseteq Y \times Z$, then $\left(\bigcup_{i\in I} R_i\right) S = \bigcup_{i\in I} R_i S$.

   Proof. (a) Note that $(w, y) \in R\left(\bigcup_{i\in I} R_i\right) \Longleftrightarrow \exists x \in X$ s.t. $(w, x) \in R$ and $(x, y) \in \bigcup_{i\in I} R_i$; and $(x, y) \in \bigcup_{i\in I} R_i \Longleftrightarrow \exists i_0 \in I$ s.t. $(x, y) \in R_i$. Then $(w, y) \in R\left(\bigcup_{i\in I} R_i\right) \Longleftrightarrow \exists i_0 \in I$ s.t. $(w, y) \in RR_i$ $\Longleftrightarrow (w, y) \in \bigcup_{i\in I} RR_i$.

   (b) Note that $(x, z) \in \left(\bigcup_{i\in I} R_i\right) S \Longleftrightarrow \exists y \in Y$ s.t. $(x, y) \in R$ and $(x, y) \in \bigcup_{i\in I} R_i$; and $(x, y) \in \bigcup_{i\in I} R_i \Longleftrightarrow \exists i_0 \in I$ s.t. $(x, y) \in R_i$. Then $(w, y) \in R\left(\bigcup_{i\in I} R_i\right) \Longleftrightarrow \exists i_0 \in I$ s.t. $(w, y) \in RR_i \Longleftrightarrow$ $(w, y) \in \bigcup_{i\in I} RR_i$. □

5. Let R_i $(1 \le i \le 3)$ be relations on $A = \{a, b, c, d, e\}$ whose Boolean matrices are

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

(a) Draw the digraphs of the relations R_1, R_2, R_3.

(b) Find the Boolean matrices for the relations

$$R_1^{-1}, \quad R_2 \cup R_3, \quad R_1R_1, \quad R_1R_1^{-1}, \quad R_1^{-1}R_1;$$

and verify that

$$R_1R_1^{-1} = R_2, \quad R_1^{-1}R_1 = R_3.$$

(c) Verify that $R_2 \cup R_3$ is an equivalence relation and find the quotient set $A/(R_2 \cup R_3)$.

Solution:

$$M_{R_1^{-1}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_{R_2 \cup R_3} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad M_{R_1^2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{R_1R_1^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M_2, \quad M_{R_1^{-1}R_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = M_3.$$

6. Let R be a relation on $\mathbb{Z}$ defined by xRy if $x + y$ is an even integer.

(a) Show that R is an equivalence relation on $\mathbb{Z}$.

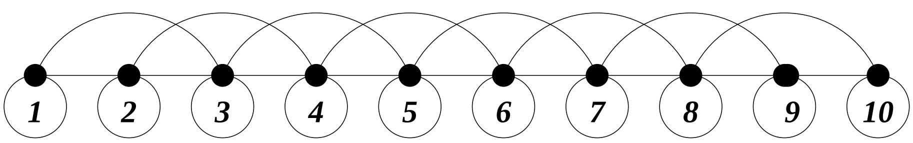
(b) Find all equivalence classes of the relation R.

Proof. (a) Since $x + x = 2x$ is even for any x, then xRx, so R is reflexive. If xRy, then $x + y$ is even. Of course, $y + x$ is even, i.e., yRx. So R is symmetric. If xRy and yRz, then $x + y$ and $y + z$ are even, so xRz. Thus R is reflexive. Therefore R is an equivalence relation.

(b) Note that xRy if and only if x and y are both odd or both even. Thus there are only two equivalence classes: the set of even integers, and the set of odd integers. □

7. Let $X = \{1, 2, \ldots, 10\}$ and let R be a relation on X such that aRb if and only if $|a - b| \le 2$. Determine whether R is an equivalence relation. Let M_R be the matrix of R. Compute M_R^8.

Solution: The following is the graph of the relation.



Then M_R^5 is a Boolean matrix all whose entries are 1. Thus M_R^8 is the same as M_R^5. □

8. A relation R on a set X is called a **preference relation** if R is reflexive and transitive. Show that $R \cap R^{-1}$ is an equivalence relation.

Proof. Obviously, $R \cap R^{-1}$ is reflexive. If $x(R \cap R^{-1})y$, then xRy and $xR^{-1}y$, i.e., $yR^{-1}x$ and yRx. Hence $y(R \cap R^{-1})x$. So $R \cap R^{-1}$ is symmetric. If $x(R \cap R^{-1})y$ and $y(R \cap R^{-1})z$, then xRy, yRz, yRx, zRy, thus xRz and zRx, i.e., xRz and $xR^{-1}z$. Therefore $x(R \cap R^{-1})z$. So $R \cap R^{-1}$ is transitive. □

9. Let n be a positive integer. The congruence relation $\sim$ of modulo n is an equivalence relation on $\mathbb{Z}$. Let $\mathbb{Z}_n$ denote the quotient set $\mathbb{Z}/\sim$. For any integer $a \in \mathbb{Z}$, we define a function

$$f_a : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n, \quad f_a([x]) = [ax].$$

   (a) Find the cardinality of the set $f_a(\mathbb{Z}_n)$.

   (b) Find all integers a such that f_a is invertible.

   Solution: (a) Let $d = \gcd(a, n)$, $a = kd$, $n = ld$. Let $x = ql + r$. Then

$$ax = kd(ql + r) = kdql + kdr = kqn + ar \equiv ar(\mod n).$$

   For $1 \le r_1 < r_2 < l$, we claim that $ar_1 \not\equiv ar_2(\mod n)$. In fact, if $ar_1 \equiv ar_2(\mod n)$, then $n|a(r_2 - r_1)$ or equivalently $l|k(r_2 - r_1)$. Since $\gcd(k, l) = 1$, we have $l|(r_2 - r_1)$. Thus $r_1 = r_2$, a contradiction. Therefore

$$f_a(\mathbb{Z}_n) = l = \frac{n}{\gcd(a, n)}.$$

   (b) Since $\mathbb{Z}_n$ is finite, then f_a is a bijection if and only if f_a is onto. However, f_a is onto if and only if $|f_a(\mathbb{Z}_n)| = n$, i.e., $\gcd(a, n) = 1$.

10. For a positive integer n, let $\phi(n)$ be the number of positive integers $x \le n$ such that $\gcd(x, n) = 1$, called **Euler's function**. Let R be the relation on $X = \{1, 2, \ldots, n\}$ defined by

$$xRy \iff x \le y,\ y|n,\ \gcd(x, y) = 1.$$

   (a) Find the cardinality $|R^{-1}(y)|$ for each $y \in X$.

   (b) Show that

$$|R| = \sum_{x|n} \phi(x).$$

   (c) Prove $|R| = n$ by showing that the function $f : R \longrightarrow X$, defined by $f(x, y) = \frac{xn}{y}$, is a bijection.

   Solution: (a) For each $y \in X$ and $y|n$, we have

$$|R^{-1}(y)| = |\{x \in X \mid x \le y, \gcd(x, y) = 1\}| = \phi(y).$$

   (b) It follows obviously that

$$|R| = \sum_{y \in X} |R^{-1}(y)| = \sum_{y \ge 1,\, y|n} |R^{-1}(y)| = \sum_{y|n} \phi(y).$$

   (c) It is clear that the function f is well-defined. For $x_1 R y_1$ and $x_2 R y_2$, if $\frac{x_1 n}{y_1} = \frac{x_2 n}{y_2}$, i.e., $\frac{x_1}{y_1} = \frac{x_2}{y_2}$, then (x_1, y_1) and (x_2, y_2) are integral proportional, say, $(x_2, y_2) = c(x_1, y_1)$. Since $\gcd(x_1, y_1) = \gcd(x_2, y_2) = 1$, then $c = 1$. We thus have $(x_2, y_2) = (x_1, y_1)$.

   On the other hand, for any $z \in X$, let $d = \gcd(z, n)$. Then

$$f\left(\frac{z}{d}, \frac{n}{d}\right) = \frac{(z/d) \cdot n}{n/d} = z.$$

   This means that f is surjective. Thus f is a bijection.

11. Let X be a set of n elements. Show that the number of equivalence relations on X is

$$\sum_{k=0}^{n} (-1)^k \sum_{l=k}^{n} \frac{(l-k)^n}{k!(l-k)!}.$$

   (Hint: Each equivalence relation corresponds to a partition. Counting number of equivalence relations is the same as counting number of partitions.)

Proof. Note that partitions of X with k parts are in one-to-one correspondent with surjective functions from X to $\{1, 2, \ldots, k\}$. By the inclusion-exclusion principle, the number of such surjective functions is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

Thus the answer is given by

$$\sum_{k=1}^{n} k! \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = \sum_{i=0}^{n} (-1)^i \sum_{k=i}^{n} \frac{(k-i)^n}{i!(k-i)!}.$$

(Note that for $k = 0$, the sum $\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = 0$.) □