Binary Relations

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1 Binary Relations

The notion of a relation between two sets of objects is quite common and intuitively clear. Let X be the set of all living human females and Y the set of all living human males. The wife-husband relation R can be defined from X to Y. Thus, if $x \in X$ and $y \in Y$, then x is related to y by the relation R if x is a wife of y, and we write xRy. To describe the relation R, we may take the collection of all ordered pairs (x, y) such that x is related to y by R; the collection of related ordered pairs is simply a subset of the product set $X \times Y$. This motivates the definition of relations.

Definition 1.1 *Let X and Y be nonempty sets. By a* **binary relation** *(or just* **relation***) from X to Y we mean a subset $R \subset X \times Y$. If $(x, y) \in R$, we say that x is related to y by R, written xRy. If $(x, y) \notin R$, we say that x is not related to y, and written $x\bar{R}y$. For $x \in X$, we define*

$$R(x) = \{y \in Y \mid xRy\} = \{y \in Y \mid (x, y) \in R\},$$

and for a subset $A \subset X$,

$$R(A) = \{y \in Y \mid \text{there exists } x \in A \text{ such that } xRy\}.$$

If $X = Y$, we say that R is a binary relation on X.

Since binary relations from X to Y are subsets of $X \times Y$, one can define intersection, union, and complement for binary relations. For a relation $R \subset X \times Y$, the **complementary relation** of R is the binary relation $\bar{R} \subset X \times Y$ defined by

$$x\bar{R}y \Leftrightarrow (x, y) \notin R;$$

and the **inverse relation** of R is the binary relation $R^{-1} \subset Y \times X$ defined by

$$yR^{-1}x \Leftrightarrow (x, y) \in R.$$

Example Consider a family A with five children, Amy, Bob, Charlie, Debbie, and Eric. We abbreviate the names to their first letters so that $A = \{a, b, c, d, e\}$.

(a) The "brother-sister" relation R_{bs} is the set

$$R_{bs} = \{(b, a), (b, d), (c, a), (c, d), (e, a), (e, d)\}.$$

(b) The "sister-brother" relation R_{sb} is the set

$$R_{sb} = \{(a,b),(a,c),(a,e),(d,b),(d,c),(d,e)\}.$$

(c) The "brother" relation R_b is the set

$$\{(b,b),(b,c),(b,e),(c,b),(c,c),(c,e),(e,b),(e,c),(e,e)\}.$$

(d) The "sister" relation R_s is the set

$$\{(a,a),(a,d),(d,a),(d,d)\}.$$

The "brother-sister" relation R_{bs} is the inverse of the "sister-brother" relation R_{sb}; that is, $R_{bs} = R_{sb}^{-1}$. The "brother or sister" relation is the union of the "brother" relation and the "sister" relation, i.e., $R_b \cup R_s$. The complementary relation of "brother or sister" relation is the "brother-sister or sister-brother" relation; that is, $\overline{R_b \cup R_s} = R_{bs} \cup R_{sb}$.

Example

(a) The graph of the equation, $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, defines a binary relation on the set $\mathbf{R}$ of real numbers; its graph is an ellipse.

(b) The relation "less than", denoted $<$, is a binary relation on $\mathbf{R}$, defined by $a < b$ if and only if a is less than b. As a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ it is also given by the set $\{(a,b) \in \mathbf{R}^2 \mid a \text{ is less than } b\}$.

(c) The relation "greater than or equal to" is a binary relation $\geq$ on $\mathbf{R}$, defined by $a \geq b$ if and only a is greater than or equal to b. As a subset of $\mathbf{R}^2$ it is given by the set $\{(a,b) \in \mathbf{R}^2 \mid a \text{ is greater than or equal to } b\}$.

(d) The divisibility relation $|$ about integers, defined by $a|b$ if and only if a divides b, is a binary relation on the set $\mathbf{Z}$ of integers.

Example A function $f : X \to Y$ can be viewed as a relation from X to Y; it is a relation $f \subset X \times Y$ such that $|f(x)| = 1$ for all $x \in X$.

Proposition 1.2 *Let R be a binary relation from X to Y. Let A and B be subsets of X.*

(a) *If $A \subset B$, then $R(A) \subset R(B)$.*

(b) *$R(A \cup B) = R(A) \cup R(B)$.*

(c) *$R(A \cap B) \subset R(A) \cap R(B)$.*

Proof. (a) For any $y \in R(A)$, there is an $x \in A$ such that xRy. Since $A \subset B$, we have $y \in R(B)$. Thus $R(A) \subset R(B)$.

(b) For any $y \in R(A \cup B)$, there is an $x \in A \cup B$ such that xRy. If $x \in A$, then $y \in R(A)$. If $x \in B$, then $y \in R(B)$. In either case, $y \in R(A) \cup R(B)$. Thus $R(A \cup B) \subset R(A \cup B)$. On the other hand, it follows from (1) that $R(A) \subset R(A \cup B)$ and $R(B) \subset R(A \cup B)$. Therefore $R(A) \cup R(B) \subset R(A \cup B)$.

(c) It follows from (1) that $R(A \cap B) \subset R(A)$ and $R(A \cap B) \subset R(B)$. Thus $R(A \cap B) \subset R(A) \cap R(B)$. □

Proposition 1.3 *Let R_1 and R_2 be relations from X to Y. If $R_1(x) = R_2(x)$ for all $x \in X$, then $R_1 = R_2$.*

Proof. If xR_1y, then $y \in R_1(x)$. Since $R_1(x) = R_2(x)$, we have $y \in R_2(x)$. Then xR_2y. A similar argument shows that if xR_2y then xR_1y. Thus $R_1 = R_2$. □

Let R be a relation on a set X. A **path of length** k in R from x to y is a finite sequence $x = x_0, x_1, \ldots, x_k = y$, beginning with x and ending with y, such that

$$x_0Rx_1,\ x_1Rx_2,\ \ldots,\ x_{k-1}Rx_k.$$

A path that begins and ends at the same vertex is called a **cycle**. For a fixed positive integer k, we define a relation R^k on X as follows:

$$xR^ky \ \Leftrightarrow\ \text{there is a path of length } k \text{ from } x \text{ to } y.$$

We may also define a relation R^∞ on X by letting

$$xR^\infty y \ \Leftrightarrow\ \text{there is some path from } x \text{ to } y.$$

The relation R^∞ is sometimes called the **connectivity relation** for R. It is clear that

$$R^\infty = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{k=1}^{\infty} R^k.$$

The **reachability relation** of R is the relation R^* on X defined by

$$xR^*y \ \Leftrightarrow\ \text{either } x = y \text{ or } xR^\infty y;$$

that is,

$$R^* = I \cup R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{k=0}^{\infty} R^k,$$

where I is the identity relation on X, defined by xIy if and only if $x = y$. We always assume that $R^0 = I$ for any relation R.

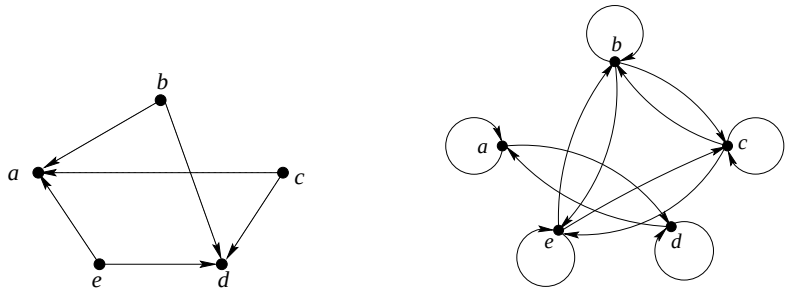
2 Representation of Relations

Binary relations are the most important relations among all relations. Ternary relations, quaternary relations, and multirelations can be studied by binary relations. We introduce two methods to represent a binary relation, one by a matrix and the other one by a directed graph.

Definition 2.1 *Let R be a binary relation from $X = \{x_1, x_2, \ldots, x_m\}$ to $Y = \{y_1, y_2, \ldots, y_n\}$. The* **matrix** *of R is an $m \times n$ matrix $M_R = [a_{ij}]$ whose (i, j)-entry is*

$$a_{ij} = \begin{cases} 1 & \text{if } (x_i, y_j) \in R \\ 0 & \text{if } (x_i, y_j) \notin R. \end{cases}$$

The matrix M_R is called a **Boolean matrix**. *If $X = Y$, M_R is a square matrix.*



(a) R_{bs} (b) $R_b \cup R_s$

Figure 1: The digraphs of relations R_{bs} and $R_b \cup R_s$.

For $m \times n$ Boolean matrices $M_1 = [a_{ij}]$ and $M_2 = [b_{ij}]$, if $a_{ij} \leq b_{ij}$ for all (i, j)-entries, we write $M_1 \leq M_2$. The matrix of the "brother-sister" relation R_{bs} on the set $A = \{a, b, c, d, e\}$ is the square matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and the matrix of the "brother or sister" relation is the square matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Another way to describe a binary relation is to draw a directed graph. Let R be a binary relation on a finite set $V = \{v_1, v_2, \ldots, v_n\}$. For each element $v_i \in V$, we draw a solid dot and name it by v_i, called a **vertex**. For two vertices v_i and v_j, if $v_i R v_j$, we draw an arrow from v_i to v_j, called a **directed edge**; if $v_i = v_j$, the arrow becomes a **directed loop**. The resulted graph is a directed graph, called the **digraph** of R, denoted $D(R)$. Note that the directed edges of a digraph may have to cross each other when drawing the digraph on a plane. However, the intersection points of directed edges are not considered to be vertices of the digraph. The **in-degree** of a vertex $v \in V$ is the number of vertices u such that $(u, v) \in R$, denoted $\text{ideg}(v)$; the **out-degree** of v is the number of vertices w such that $(v, w) \in R$, denoted $\text{odeg}(v)$. If R is a relation from X to Y, we define

$$\begin{aligned} \text{odeg}(x) &= |R(x)| \quad \text{for all } x \in X, \\ \text{ideg}(y) &= |R^{-1}(y)| \quad \text{for all } y \in Y. \end{aligned}$$

The digraphs of the "brother-sister" relation R_{bs} and the "brother or sister" relation $R_b \cup R_s$ are demonstrated in the following.

Proposition 2.2 *For the digraph $D(R)$ of a binary relation R on V,*

$$\sum_{v \in V} \text{ideg}(v) = \sum_{v \in V} \text{odeg}(v) = |R|.$$

If R is a relation from X to Y, then

$$\sum_{x\in X} \mathrm{odeg}(x) = \sum_{y\in Y} \mathrm{ideg}(y) = |R|.$$

Proof. Trivial. □

3 Composition of Relations

Definition 3.1 *Let R be a relation from X to Y, and S a relation from Y to Z. The* **composition** *of R and S is a relation $S \circ R$ from X to Z, defined by*

$$x(S \circ R)z \;\Leftrightarrow\; \text{there is an element } y \in Y \text{ such that } xRy \text{ and } ySz.$$

If $X = Y$, then R is a relation on X; we have $R^2 = R \circ R$ and $R^k = R^{k-1} \circ R$ for $k \geq 2$.

Note that in the composition of R and S we consider R as the first relation and S the second, and the notation $S \circ R$ is backward. However, many people use $R \circ S$ as a name for what we have called $S \circ R$. Such usage is inconsistent with the notation for functional composition and causes some confusion. To avoid misunderstanding and for aesthetic reason, we often write RS for $S \circ R$.

Example For the "brother-sister relation," "sister-brother relation," "brother relation," and "sister relation" on $A = \{a, b, c, d, e\}$, we have

$$\begin{array}{lll} R_{bs}R_{sb} = R_b, & R_{sb}R_{bs} = R_s, & R_{bs}R_s = R_{bs}, \\ R_{bs}R_{bs} = \emptyset, & R_bR_b = R_b, & R_bR_s = \emptyset. \end{array}$$

Let X_1, X_2, …, X_n, X_{n+1} be nonempty sets. Let R_i be a relation from X_i to X_{i+1}, $1 \leq i \leq n$. We define a relation $R_1R_2\cdots R_n$ from X_1 to X_{n+1} by

$$xR_1R_2\cdots R_ny$$

if and only if there is a sequence $x = x_1, x_2, \ldots, x_n, x_{n+1} = y$ such that

$$x_1R_1x_2,\ x_2R_2x_3,\ \ldots,\ x_nR_nx_{n+1}.$$

Theorem 3.2 *Let R_1 be a relation from X_1 to X_2, R_2 a relation from X_2 to X_3, and R_3 a relation from X_3 to X_4. Then $R_1(R_2R_3)$ and $(R_1R_2)R_3$ are relations from X_1 to X_4, and*

$$R_1R_2R_3 = R_1(R_1R_3) = (R_1R_2)R_3.$$

Proof. For any $x \in X_1$ and $y \in X_4$, we have

$$\begin{array}{rcl} xR_1(R_2R_3)y & \Leftrightarrow & \exists x_2 \in X_2 \text{ s.t. } xR_1x_2,\ x_2R_2R_3y \\ & \Leftrightarrow & \exists x_2 \in X_2 \text{ s.t. } xR_1x_2;\ \exists x_3 \in X_3 \text{ s.t. } x_2R_2x_3,\ x_3R_3y \\ & \Leftrightarrow & \exists x_2 \in X_2, x_3 \in X_3 \text{ s.t. } xR_1x_2,\ x_2R_2x_3,\ x_3R_3y \\ & \Leftrightarrow & xR_1R_2R_3y. \end{array}$$

Similarly, $x(R_1R_2)R_3y \Leftrightarrow xR_1R_2R_3y$. □

Proposition 3.3 *Let R be a relation from X to Y, R_i $(i \in I)$ a family of relations from Y to Z, and S a relation from Z to W. Then*

(a) $R\left(\bigcup_{i\in I} R_i\right) = \bigcup_{i\in I} RR_i$;

(b) $\left(\bigcup_{i\in I} R_i\right) S = \bigcup_{i\in I} R_i S$.

Proof. (a) For any $x(R \cup_i R_i)z$, there exists $y \in Y$ such that xRy and $y(\cup_i R_i)z$. Then there is one j such that yR_jz. Thus $x(RR_i)z$, and so $x(\bigcup_i RR_i)z$. Conversely, for any $x(\cup_i RR_i)z$, there is one i such that $x(RR_i)z$. Then there exists $y \in Y$ such that xRy and yR_iz. Of course, $y(\cup_i R_i)z$. Thus $x(R \cup_i R_i)z$. The proof for (b) is similar. □

For the convenience of representing composition of relations, we introduce two operations $\wedge$ and $\vee$ on real numbers. For $a, b \in \mathbf{R}$, define

$$a \wedge b = \min\{a, b\},$$

$$a \vee b = \max\{a, b\}.$$

Lemma 3.4 *For $a, b, c \in \mathbf{R}$,*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Proof. We only prove the first formula. The second one is similar.

Case 1: $b \leq c$. If $a \geq c$, then the left side is $a \wedge (b \vee c) = a \wedge c = c$. The right side is $(a \wedge b) \vee (a \wedge c) = b \vee c = c$. If $b \leq a \leq c$, then the left side is $a \wedge (b \vee c) = a \wedge c = a$. The right side is $(a \wedge b) \vee (a \wedge c) = b \vee a = a$. If $a \leq b \leq c$, then the left side is $a \wedge (b \vee c) = a \wedge c = a$. The right side is $(a \wedge b) \vee (a \wedge c) = a \vee a = a$.

Case 2: $b \geq c$. If $a \leq c$, then $a \wedge (b \vee c) = a \wedge c = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee a = a$. If $b \geq a \geq c$, then $a \wedge (b \vee c) = a \wedge c = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee c = a$. If $a \geq b$, then $a \wedge (b \vee c) = a \wedge b = b$ and $(a \wedge b) \vee (a \wedge c) = b \vee c = b$. □

For real numbers $a_1, a_2, \ldots, a_n$, we define

$$\bigwedge_{i=1}^{n} a_i = \min\{a_1, a_2, \ldots, a_n\},$$

$$\bigvee_{i=1}^{n} a_i = \max\{a_1, a_2, \ldots, a_n\}.$$

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ an $n \times p$ matrix. The **Boolean multiplication** of A and B is an $m \times p$ matrix $A * B = [c_{ik}]$, whose (i, k)-entry is

$$c_{ik} = \bigvee_{j=1}^{n} (a_{ij} \wedge b_{jk}).$$

Theorem 3.5 *Let R be a relation from $X = \{x_1, \ldots, x_m\}$ to $Y = \{y_1, \ldots, y_n\}$, and let S be a relation from Y to $Z = \{z_1, \ldots, z_p\}$. Let M_R be the matrix of R, M_S the matrix of S, and M_{RS} the matrix of RS. Then*

$$M_{RS} = M_R * M_S.$$

Proof. Write $M_R = [a_{ij}]$, $M_S = [b_{jk}]$, $M_R * M_S = [c_{ik}]$, and $M_{RS} = [d_{ik}]$. It suffices to show that $c_{ik} = d_{ik}$ for each (i,k)-entry of the matrices. In fact, if $c_{ik} = 1$, it forces that $a_{ij} \wedge b_{jk} = 1$ for at least one j, say j_0. Then $a_{ij_0} = b_{j_0k} = 1$. This means that $x_i R y_{j_0}$ and $y_{j_0} S z_k$. Thus $x_i RS z_k$, and so $d_{ik} = 1$. If $c_{ik} = 0$, then $a_{ij} \wedge b_{jk} = 0$ for all j's; that is, there is no $y_j \in Y$ such that $x_i R y_j$ and $y_j S z_k$. Thus x_i is not related to z_k by RS. Therefore $d_{ik} = 0$. This completes the proof of $c_{ik} = d_{ik}$. □

4 Special Relations

Most of time we are interested in some special relations satisfying certain properties. For instance, the "less than" relation on the set of real numbers satisfies the so-called transitive property: if $a < b$ and $b < c$, then $a < c$.

Definition 4.1 *A binary relation R on a set X is called*

(a) **reflexive** *if xRx for all x in X;*

(b) **symmetric** *if xRy implies yRx;*

(c) **transitive** *if xRy and yRz imply xRz.*

The relation R is called an **equivalence relation** *if it is reflexive, symmetric, and transitive; and in this case, if xRy, we say that x and y are* **equivalent**.

The relation $I = I_X = \{(x,x) \mid x \in X\}$ is called the **identity relation**, *and X^2 is called the* **complete relation** *on X.*

Example Many family relations are binary relations on the set of human beings.

(a) The brother relation R_b: $xR_by \Leftrightarrow x$ and y are both males and have the same parents. (symmetric and transitive)

(b) The sister relation R_s: $xR_sy \Leftrightarrow x$ and y are both females and have the same parents. (symmetric and transitive)

(c) The brother-sister relation R_{bs}: $xR_{bs}y \Leftrightarrow x$ is male, y is female, x and y have the same parents.

(d) The sister-brother relation R_{sb}: $xR_{sb} \Leftrightarrow x$ is female, y is male, and x and y have the same parents.

(e) The generalized brother relation R'_b: $xR'_by \Leftrightarrow x$ and y are both males and have the same father or the same mother. (symmetric)

(f) The generalized sister relation R'_s: $xR'_sy \Leftrightarrow x$ and y are both females and have the same father or mother. (symmetric)

(g) The relation R: $xRy \Leftrightarrow x$ and y have the same parents. (reflexive, symmetric, and transitive; equivalence relation)

(h) The relation R': $xR'y \Leftrightarrow x$ and y have the same father or the same mother. (reflexive and symmetric)

Example

(a) The "less than" relation $<$ on the set of real numbers is a transitive relation.

(b) The "less than or equal to" relation $\leq$ on the set of real numbers is a reflexive and transitive relation.

(c) The divisibility relation on the set of positive integers is a reflexive and transitive relation.

(d) Given a positive integer n; the **congruence of modulo** n is a relation $\equiv_n$ on $\mathbf{Z}$, defined by $a \equiv_n b$ if and only if $b - a$ is a multiple of n. The standard notation for $a \equiv_n b$ is $a \equiv b \pmod{n}$. The relation $\equiv_n$ is an equivalence relation on $\mathbf{Z}$.

Theorem 4.2 *Let R be a relation on a set X. Then*

(a) *R is reflexive $\Leftrightarrow I \subset R \Leftrightarrow$ all diagonal entries of M_R are 1.*

(b) *R is symmetric $\Leftrightarrow R = R^{-1} \Leftrightarrow M_R$ is a symmetric matrix.*

(c) *R is transitive $\Leftrightarrow R^2 \subset R \Leftrightarrow M_R^2 \leq M_R$.*

Proof. (a) and (b) are trivial.

(c) "R is transitive $\Rightarrow R^2 \subset R$." For any $(x, y) \in R^2$, there exists $z \in X$ such that $(x, z) \in R$ and $(z, y) \in R$. Since R is transitive, we have $(x, y) \in R$. So $R^2 \subset R$.

"$R^2 \subset R \Rightarrow R$ is transitive." For $(x, z) \in R$ and $(z, y) \in R$, we have $(x, y) \in R^2 \subset R$. Then $(x, y) \in R$. So R is transitive.

Note that for any relations R and S on X, $R \subset S$ if and only if $M_R \leq M_S$. Also note that the matrix M_{R^2} of R^2 is M_R^2. We thus have that $R^2 \subset R$ if and only if $M_R^2 \leq M_R$. □

5 Equivalence Relations and Partitions

The most important relations among binary relations are equivalence relations. We will see that an equivalence relation on a set X will partition X into disjoint equivalence classes.

Example Consider the congruence relation $\equiv_3$ on $\mathbf{Z}$. For each $a \in \mathbf{Z}$, define

$$[a] = \{b \in \mathbf{Z} \mid a \equiv_3 b\} = \{b \in \mathbf{Z} \mid a \equiv b \pmod{3}\}.$$

It is clear that $\mathbf{Z}$ is partitioned into three disjoint subsets

$$\begin{aligned} [0] &= \{0, 3, \pm 6, \pm 9, \ldots\} = \{3k \mid k \in \mathbf{Z}\}, \\ [1] &= \{1, 1 \pm 3, 1 \pm 6, 1 \pm 9, \ldots\} = \{3k + 1 \mid k \in \mathbf{Z}\}, \\ [2] &= \{2, 2 \pm 3, 2 \pm 6, 2 \pm 9, \ldots\} = \{3k + 2 \mid k \in \mathbf{Z}\}. \end{aligned}$$

Moreover, $[3k] = [0]$, $[3k + 1] = [1]$, and $[3k + 2] = [2]$ for all $k \in \mathbf{Z}$.

Theorem 5.1 *Let $\sim$ be an equivalence relation on a set X. For each x of X, let $[x] = \{y \in X \mid x \sim y\}$. Then*

(a) *$[x] \neq \emptyset$ for any $x \in X$,*

(b) $[x] = [y]$ *if* $x \sim y$,

(c) $[x] \cap [y] = \emptyset$ *if* $x \not\sim y$,

(d) $X = \bigcup_{x \in X} [x]$.

Each subset $[x]$ *is called an* **equivalence class**, *and* x *is called a* **representative** *of* $[x]$.

Proof. (a) Each $[x]$ is obviously nonempty because $\sim$ is reflexive.

(b) For any $z \in [x]$, we have $x \sim z$ by definition of $[x]$. Since $x \sim y$, we have $y \sim x$ by the symmetric property of $\sim$. Then $y \sim x$ and $x \sim z$ imply that $y \sim z$ by transitivity of $\sim$. Thus $z \in [y]$ by definition of $[y]$; that is, $[x] \subset [y]$. Since $\sim$ is symmetric, we have $[y] \subset [x]$. Therefore $[x] = [y]$.

(c) Suppose $[x] \cap [y]$ is not empty, say $z \in [x] \cap [y]$. Then $x \sim z$ and $y \sim z$. By symmetry of $\sim$, we have $z \sim y$. Thus $x \sim y$ by transitivity of $\sim$, a contradiction.

(d) This is obvious because $x \in [x]$ for any $x \in X$. □

Definition 5.2 *A* **partition** *of a nonempty set* X *is a collection* $\mathcal{P} = \{A_j \mid j \in J\}$ *of nonempty subsets of* X *such that*

(a) $A_i \cap A_j = \emptyset$ *if* $i \neq j$,

(b) $X = \bigcup_{j \in J} A_j$.

Each subset A_j *is called a* **block** *of the partition* $\mathcal{P}$.

Theorem 5.3 *Let* $\mathcal{P}$ *be a partition of a set* X. *Let* $R_{\mathcal{P}}$ *be the relation on* X, *defined by*

$$xR_{\mathcal{P}}y \Leftrightarrow \text{there exists a block } A_j \in \mathcal{P} \text{ such that } x, y \in A_j.$$

Then $R_{\mathcal{P}}$ *is an equivalence relation on* X, *called the* **equivalence relation determined by** $\mathcal{P}$.

Proof. (a) For each $x \in X$, there exits one A_j such that $x \in A_j$. Then by definition of $R_{\mathcal{P}}$, $xR_{\mathcal{P}}x$. So $R_{\mathcal{P}}$ is reflexive. (b) If $xR_{\mathcal{P}}y$, then there is one A_j such that $x, y \in A_j$. Again by definition of $R_{\mathcal{P}}$, $yR_{\mathcal{P}}x$. Thus $R_{\mathcal{P}}$ is symmetric. (c) If $xR_{\mathcal{P}}y$ and $yR_{\mathcal{P}}z$, then there exist A_i and A_j such that $x, y \in A_i$ and $y, z \in A_j$. Obviously, $z \in A_i \cap A_j$. Since $\mathcal{P}$ is a partition. It forces that $A_i = A_j$. Thus $xR_{\mathcal{P}}z$. This shows that $R_{\mathcal{P}}$ is transitive. □

For an equivalence relation R on a set X, the collection $\mathcal{P}_R = \{[x] : x \in X\}$ of equivalence classes of R forms a partition of X, called the **quotient set** of X under R. Let $E(X)$ denote the set of all equivalence relations on X and $P(X)$ the set of all partitions of X. Define functions

$$f : E(X) \to P(X) \quad \text{by} \quad f(R) = \mathcal{P}_R,$$
$$g : P(X) \to E(X) \quad \text{by} \quad g(\mathcal{P}) = R_{\mathcal{P}}.$$

Then f and g satisfy the following properties.

Theorem 5.4 *For any equivalence relation* R *on* X *and any partition* $\mathcal{P}$ *of* X,

$$(g \circ f)(R) = R \quad \text{and} \quad (f \circ g)(\mathcal{P}) = \mathcal{P}.$$

In other words, f *and* g *are inverse of each other.*

Proof. Note that $(g \circ f)(R) = g(f(R))$ and $(f \circ g)(\mathcal{P}) = f(g(\mathcal{P}))$. We have

$$x[g(f(R))(R)]y \Leftrightarrow \exists A \in f(R) \text{ s.t. } x, y \in A \Leftrightarrow xRy;$$

$$A \in f(g(\mathcal{P})) \Leftrightarrow \exists x \text{ s.t. } A = g(\mathcal{P})(x) \Leftrightarrow A \in \mathcal{P}.$$

□

Example 5.5 Let $\mathbf{Z}_+$ be the set of positive integers. Define the relation $\sim$ on $\mathbf{Z} \times \mathbf{Z}_+$ by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

Is $\sim$ an equivalence relation? If Yes, what are the equivalence classes?

Let R be a relation on a set X. The **reflexive closure** is a reflexive relation $r(R)$ on X such that

(a) $R \subset r(R)$;

(b) if R' is a reflexive relation on X and $R \subset R'$ then $r(R) \subset R'$.

The **symmetric closure** of R is a symmetric relation $s(R)$ on X such that

(a) $R \subset s(R)$;

(b) if R' is a symmetric relation on X and $R \subset R'$ then $s(R) \subset R'$.

The **transitive closure** of R is a transitive relation $t(R)$ on X such that

(a) $R \subset t(R)$;

(b) if R' is a transitive relation on X and $R \subset R'$ then $t(R) \subset R'$.

Obviously, the reflexive, symmetric, and transitive closures of R need to be unique respectively.

Theorem 5.6 *For any relation R on a set X,*

(a) $r(R) = R \cup I$;

(b) $s(R) = R \cup R^{-1}$;

(c) $t(R) = R^\infty = \bigcup_{k=1}^\infty R^k$.

Proof. (a) and (b) are obvious. (c) Note that $R \subset \bigcup_{k=1}^\infty R^k$ and

$$\left(\bigcup_{i=1}^\infty R^i\right)\left(\bigcup_{j=1}^\infty R_j\right) = \bigcup_{i,j=1}^\infty R^i R^j = \bigcup_{i,j=1}^\infty R^{i+j} = \bigcup_{k=2}^\infty R^k \subset \bigcup_{k=1}^\infty R^k.$$

This shows that $\bigcup_{k=1}^\infty R^k$ is transitive and $R \subset \bigcup_{k=1}^\infty R^k$. Since any transitive relation which contains R must contain R^k for all positive integers k. Then $\bigcup_{k=1}^\infty R^k$ is the transitive closure of R. □

Theorem 5.7 *Let R be a relation on a set X of n elements. Then*

$$t(R) = \bigcup_{k=1}^{n-1} R^k.$$

Proof. It suffices to show that $R^l \subset \bigcup_{k=1}^{n-1} R^k$ for all $l \geq n$. For any $(x, y) \in R^l$, there exist $x_1, \ldots, x_{l-1} \in X$ such that all (x, x_1), (x_1, x_2), $\ldots$, (x_{l-1}, y) belong to R. Since $l \geq n$, at least two elements of $x = x_0$, x_1, x_2, $\ldots$, x_{l-1} ,$x_l = y$ must be the same, say $x_i = x_j$ with $i < j$. Then (x_0, x_1), $\ldots$, (x_{i-1}, x_i), (x_j, x_{j+1}), $\ldots$, $(x_{l-1}, x_l) \in R$ imply that $(x, y) = (x_0, x_l) \in R^{n+i-j}$ and $l + i - j < l$. It follows by induction that $R^l \subset \bigcup_{k=1}^{n-1} R^k$. □

Proposition 5.8 *Let R be a relation on a set X. Then $I \cup t(R \cup R^{-1})$ is an equivalence relation. In particular, if R is reflexive and symmetric, $t(R)$ is an equivalence relation.*

Let R be a relation on a set X. The **reachability relation** of R is a relation R^* on X, defined by xR^*y if and only if either $x = y$ or there exist $x_1, x_2, \ldots, x_k$ such that $(x, x_1), (x_1, x_2), \ldots, (x_k, y) \in R$; that is, $R^* = I \cup t(R)$.

Theorem 5.9 *Let R be a relation on a set X. Let M and M^* be the matrices of R and R^* respectively. If $|X| = n$, then*

$$M^* = I \vee M \vee M^2 \vee \cdots \vee M^{n-1}.$$

Moreover, if R is symmetric, then $M^* = I \vee M^{n-1}$; if R is reflexive and symmetric, then $M^* = M^{n-1}$.

Let R be a relation on $X = \{x_1, \ldots, x_n\}$. If $y_0, y_1, \ldots, y_m$ is a path in R, the vertices $y_1, \ldots, y_{m-1}$ are called **interior vertices** of the path. For each $1 \leq k \leq n$, we define a Boolean matrix $W_k = [w_{ij}]$, where $w_{ij} = 1$ if there is a path in R from x_i to x_j whose interior vertices are contained in $\{x_1, \ldots, x_k\}$. Since the interior vertices of any path in R is contained in $X = \{x_1, \ldots, x_n\}$, the (i, j)-entry of W_n is equal to 1 if there is a path in R from x_i to x_j; that is, $W_n = M_{R^\infty}$. We set $W_0 = M_R$. Then we have a sequence of Boolean matrices

$$W_0 = M_R,\ W_1,\ W_2,\ \ldots, W_n.$$

We will give an algorithm, called **Warshall's algorithm**, to compute W_k from W_{k-1}.

Let $W_{k-1} = [s_{ij}]$ and $W_k = [t_{ij}]$. If $t_{ij} = 1$, then there must be a path

$$x_i = y_0, y_1, \ldots, y_m = x_j$$

from x_i to x_j whose interior vertices are contained in $\{x_1, \ldots, x_k\}$. We may assume that the interior vertices $y_1, \ldots, y_{m-1}$ are distinct. If x_k is not an interior vertex of this path, then all interior vertices must be actually contained in $\{x_1, \ldots, x_{k-1}\}$, so $s_{ij} = 1$. If x_k is an interior of the path, say $x_k = y_l$, we then have two subpaths

$$x_i = y_0, y_1, \ldots, y_l = x_k \quad \text{and} \quad x_k = y_l, y_{l+1}, \ldots, y_m = x_j$$

whose interior vertices are both contained in $\{x_1, \ldots, x_{k-1}\}$, so $s_{ik} = 1$ and $s_{kj} = 1$. Thus

$$t_{ij} = 1 \Leftrightarrow \begin{cases} s_{ij} = 1 \text{ or} \\ s_{ik} = 1 \text{ and } s_{kj} = 1. \end{cases}$$

Warshall's Algorithm Working on the Boolean matrix W_{k-1} to produce W_k.

(a) If W_{k-1} has 1 in (i,j)-entry, so is W_k; keep 1 there.

(b) If W_{k-1} has 0 in (i,j)-entry, then check the column k of W_{k-1} at position i, and the row k of W_{k-1} at position j; if both positions are 1, add 1 to the (i,j)-entry of W_{k-1}; otherwise, keep 0 there.

Example Consider the relation R on $A = \{1,2,3,4,5\}$ defined the Boolean matrix

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

By Warshall's algorithm, we have

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \to W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1* \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \to W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \to$$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1* & 1 & 1 & 0 & 1* \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1* & 0 & 1 & 1 & 1* \end{bmatrix} \to W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \to W_5 = \begin{bmatrix} 1* & 0 & 1* & 1* & 1 \\ 1 & 1 & 1 & 1* & 1 \\ 1 & 0 & 1* & 1* & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Definition 5.10 *A binary relation R on a set X. R is called*

(a) **asymmetric** *if xRy implies $y\bar{R}x$;*

(b) **antisymmetric** *if xRy and yRx imply $x = y$.*

Exercises

1. Let R be a binary relation from X to Y. Let A and B be subsets of X.
   (a) If $A \subset B$, then $R(A) \subset R(B)$.
   (b) $R(A \cup B) = R(A) \cup R(B)$.
   (c) $R(A \cap B) \subset R(A) \cap R(B)$.
2. Let R_1 and R_2 be relations from X to Y. If $R_1(x) = R_2(x)$ for all $x \in X$, then $R_1 = R_2$.
3. Let $a, b, c \in \mathbf{R}$. Then
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$
4. Let R be a relation from X to Y, R_i $(i \in I)$ a family of relations from Y to Z, and S a relation from Z to W. Then

(a) $R\left(\bigcup_{i\in I} R_i\right) = \bigcup_{i\in I} RR_i$;

(b) $\left(\bigcup_{i\in I} R_i\right) S = \bigcup_{i\in I} R_i S$.

5. Let R_i $(1 \le i \le 3)$ be relations on $A = \{a, b, c, d, e\}$, whose Boolean matrices are given by

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

respectively.

(a) Draw the digraphs of the relations R_1, R_2, R_3.

(b) Find the Boolean matrices for the relations R_1^{-1}, $R_2 \cup R_3$, R_1R_1, $R_1R_1^{-1}$, $R_1^{-1}R_1$, $R_1 \cup R_1^{-1}$; and verify that $R_1R_1^{-1} = R_2$, $R_1^{-1}R_1 = R_3$.

(c) Verify that $R_2 \cup R_3$ is an equivalence relation and find the quotient set $A/(R_2 \cup R_3)$.

6. Let R be a relation on $\mathbf{Z}$ defined by xRy if $x + y$ is an even integer.

(a) Show that R is an equivalence relation on $\mathbf{Z}$.

(b) Find all equivalence classes of the relation R.

7. Let $X = \{1, 2, \ldots, 10\}$ and let R be a relation on X such that aRb if and only if $|a - b| \le 2$. Determine whether R is an equivalence relation. Let M_R be the matrix of R; compute M_R^8.

8. A relation R on a set X is called a **preference relation** if R is reflexive and transitive. Show that $R \cup R^{-1}$ is an equivalence relation.

9. For a positive integer n, let $\phi(n)$ be the number of positive integers $x \le n$ such that $\gcd(x, n) = 1$, called **Euler's function**. Let R be the relation on $X = \{1, 2, \ldots, n\}$, defined by

$$xRy \text{ if and only if } x \le y,\ y|n, \text{ and } \gcd(x, y) = 1.$$

(a) For each $y \in X$, find the cardinality $|R^{-1}(y)|$.

(b) Show that

$$|R| = \sum_{x|n} \phi(x).$$

(c) Show that $|R| = n$ by proving that the function $f : R \to X$, defined by $f(x, y) = \frac{xn}{y}$, is a bijection.

10. Let X be a set of n elements. Show that the number of equivalence relations on X is

$$\sum_{k=0}^{n} (-1)^k \sum_{l=k}^{n} \frac{(l-k)^n}{k!(l-k)!}.$$

(Hint: equivalence relations are in one-to-one correspondent with partitions.)