Armed with the method of this example, we now proceed to the proof of Proposition 10.1. For this proof we shall require a couple of preliminary results.

LEMMA 10.1
(a) If $0 < q < \frac{1}{2}$ then $(1-q)^n \geq 1 - nq$ for all $n \geq 1$.
(b) If $0 < q < \frac{1}{2}$ then $(1+q)^n \leq 1 + 2^n q$ for all $n \geq 1$.

PROOF Part (a) follows immediately from Example 8.2 in Chapter 8. And part (b) is Exercise 9 at the end of Chapter 8 (which you have, of course, done perfectly). ∎

By the way, this result is called a "lemma" rather than a proposition, as it is just a helpful result to be used in the proof of another, more important result, and does not have very much interest of its own; the word "lemma" is used for results of this nature. The same term applies to the next result.

LEMMA 10.2
Let $y > 0$ and $0 < \alpha < \frac{1}{2}y$. Then for all $n \geq 1$, the following are true:

(a) $(y-\alpha)^n \geq y^n - ny^{n-1}\alpha$
(b) $(y+\alpha)^n \leq y^n + 2^n y^{n-1}\alpha$.

PROOF For (a), observe that

$$(y-\alpha)^n = y^n\left(1 - \frac{\alpha}{y}\right)^n .$$

As $0 < \frac{\alpha}{y} < \frac{1}{2}$, by Lemma 10.1 the right-hand side above is greater than or equal to

$$y^n\left(1 - \frac{n\alpha}{y}\right) = y^n - ny^{n-1}\alpha ,$$

giving part (a).

To prove (b), note that by Lemma 10.1 we have $(1+\frac{\alpha}{y})^n \leq 1 + 2^n\frac{\alpha}{y}$, whence

$$(y+\alpha)^n = y^n\left(1 + \frac{\alpha}{y}\right)^n \leq y^n\left(1 + 2^n\frac{\alpha}{y}\right) = y^n + 2^n y^{n-1}\alpha . \quad ∎$$

Proof of Proposition 10.1 Let x be a positive real number, and n a positive integer. We wish to find a real number y such that $y^n = x$. Motivated by the previous example, let us define S to be the following set of real numbers:

$$S = \{s \mid s \in \mathbb{R}, s^n < x\} .$$

Certainly S has an upper bound: for if $x \geq 1$ then $x^n \geq x$, hence $s^n < x \leq x^n$ for all $s \in S$, so $s < x$ for all $s \in S$ and x is an upper bound for S; and if $x < 1$ then $s^n < x < 1$ for all $s \in S$, and so 1 is an upper bound. Therefore, by Theorem 10.1, S has a least upper bound; let

$$y = \mathrm{LUB}(S) .$$

We shall show that $y^n = x$, so that y is the real number we are seeking. As in the example above, we shall do this by contradiction. So suppose that $y^n \neq x$. Then either $y^n < x$ or $y^n > x$. We divide the argument into two cases accordingly.

Case 1 Assume first that $y^n < x$. Our strategy in this case is as in the example — to find a small $\alpha > 0$ such that $(y+\alpha)^n < x$ still; if we do this, we have $y + \alpha \in S$, whereas y is an upper bound for S, which is a contradiction.

To find α, observe that by Lemma 10.2(b),

$$(y+\alpha)^n < x \Leftarrow y^n + 2^n y^{n-1}\alpha < x \text{ and } 0 < \alpha < \frac{y}{2}$$
$$\Leftarrow x - y^n > 2^n y^{n-1}\alpha \text{ and } 0 < \alpha < \frac{y}{2} .$$

Since $x - y^n > 0$ in this case, we can choose α such that $0 < \alpha < \frac{y}{2}$ and $\alpha < \frac{x-y^n}{2^n y^{n-1}}$. Then the above implications show that $(y+\alpha)^n < x$, giving a contradiction as explained above.

Case 2 Now assume that $y^n > x$. In this case our strategy is to find a small number $\beta > 0$ such that $(y-\beta)^n > x$ still. If we do this, then for $s \in S$ we have $s^n < x < (y-\beta)^n$, hence $s < y - \beta$, and so $y - \beta$ is an upper bound for S. However, y is the LUB of S, so this is a contradiction.

To find β, note that by Lemma 10.2(a),

$$(y-\beta)^n > x \Leftarrow y^n - ny^{n-1}\beta > x \text{ and } 0 < \beta < \frac{y}{2}$$
$$\Leftarrow y^n - x > ny^{n-1}\beta \text{ and } 0 < \beta < \frac{y}{2} .$$

Since $y^n > x$ we can choose β such that $0 < \beta < \frac{y}{2}$ and $\beta < \frac{y^n - x}{ny^{n-1}}$. Then $(y-\beta)^n > x$, giving a contradiction as explained above.

We have reached a contradiction in both Cases 1 and 2, so we conclude that $y^n = x$. Finally, $y > 0$ since S contains some positive numbers and y is an upper bound for S. And y is unique, since if $y^n = z^n = x$ with $z > 0$, then $y = z$: for otherwise, either $y > z$ or $y < z$, implying that either $y^n < z^n$ or $y^n > z^n$, neither of which is true.

This completes the proof of Proposition 10.1.