king rejoinders, "You are correct, but you will be beheaded anyway." The executioner's axe is just coming down when Ivor wakes up. He wonders whether his answer to the king was really correct.

Prove that Ivor was indeed correct.

Chapter 13

More on Prime Numbers

As you are probably beginning to appreciate, the prime numbers are fundamental to our understanding of the integers. In this chapter we will discuss a few basic results concerning the primes, and also hint at the vast array of questions, some solved, some unsolved, in current research into prime numbers.

The first few primes are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots$$

It is quite simple to carry on a long way with this list, particularly if you have a computer at hand. How would you do this? The easiest way is probably to test each successive integer n for primality, by checking, for each prime $p \leq \sqrt{n}$, whether p divides n (such primes p will of course already be in your list). If none of these primes p divides n, then n is prime — see Exercise 2 at the end of the chapter. Some more sophisticated methods for primality testing will be discussed at the end of Chapter 15.

Probably the first and most basic question to ask is: Does this list ever stop? In other words, is there a *largest* prime number, or does the list of primes go on forever? The answer is provided by the following famous theorem of Euclid (300 BC).

THEOREM 13.1
There are infinitely many prime numbers.

PROOF This is one of the classic proofs by contradiction. Assume the result is false, i.e., there are only finitely many primes. This means that we can make a finite list

$$p_1, p_2, p_3, \ldots, p_n$$

of *all* the prime numbers. Now define a positive integer

$$N = p_1 p_2 p_3 \cdots p_n + 1 .$$