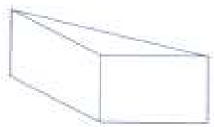
(3) *Triangular prism*



This has 6 vertices, 9 edges and 5 faces.

(4) *n-prism* This is like the triangular prism, except that its top and bottom faces are n-sided polygons rather than triangles. It has $2n$ vertices, $3n$ edges and $n+2$ faces.

Let us collect the numbers of vertices, edges and faces for the above examples in a table. Denote these numbers by V, E and F, respectively.

	V	E	F
(1)	8	12	6
(2)	4	6	4
(3)	6	9	5
(4)	$2n$	$3n$	$n+2$

Can you see a relationship between these numbers that holds in every case? You probably can — it is

$$V - E + F = 2 .$$

This is Euler's famous formula, and we shall show that it holds in general for all *convex* polyhedra: a polyhedron is convex if, whenever we choose two points on its surface, the straight line joining them lies entirely within the polyhedron.

All of the above examples are convex polyhedra. However, if we for example take a cube and remove a smaller cube from its interior, we get a polyhedron that is not convex; for this polyhedron, in fact, $V = 16, E = 24, F = 12$, so $V - E + F = 4$ and the formula fails.

Here then is Euler's formula.

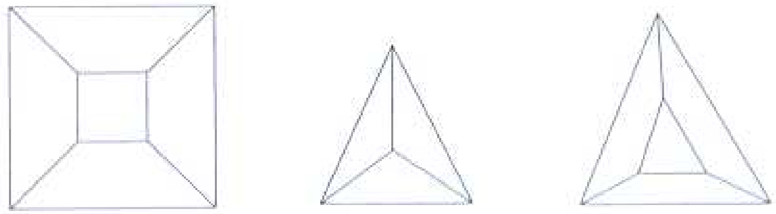
THEOREM 9.1
For a convex polyhedron with V vertices, E edges and F faces, we have

$$V - E + F = 2 .$$

As I said, we shall prove this result by induction. So somehow we have to design a suitable statement $P(n)$ to try to prove by induction. What on earth should $P(n)$ be?

Before going into this, let us first translate the problem from one about objects in 3-dimensional space to one about objects in the plane. Take a convex polyhedron as in the theorem, and choose one face of it. Regard this face as a

window, put your eye very close to the window, and draw on the window pane the vertices and edges you can see through the window. The result is a figure in the plane with straight edges, vertices and faces. For example, here is what we would draw for the cube, the tetrahedron and the triangular prism:

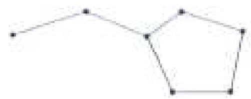


(The outer edges enclose the window.)

The resulting figure in the plane has V vertices, E edges and $F-1$ faces (we lose one face, since the window is no longer a face). It is a "connected plane graph," in the sense of the following definition.

DEFINITION *A* plane graph *is a figure in the plane consisting of a collection of points (vertices), and some edges joining various pairs of these points, with no two edges crossing each other. A plane graph is* connected *if we can get from any vertex of the graph to any other vertex by going along a path of edges in the graph.*

For example, here is a connected plane graph:



It has 7 vertices, 7 edges and 1 face.

THEOREM 9.2
If a connected plane graph has v vertices, e edges and f faces, then

$$v - e + f = 1 .$$

Theorem 9.2 easily implies Euler's theorem 9.1: for if we have a convex polyhedron with V vertices, E edges and F faces, then as explained above we get a connected plane graph with V vertices, E edges and $F-1$ faces. If we knew Theorem 9.2 was true, we could then deduce that $V - E + (F-1) = 1$, hence $V - E + F = 2$, as required for Euler's theorem.

So we need to prove Theorem 9.2.