PROPOSITION 8.1
Every positive integer greater than 1 is equal to a product of prime numbers.

In the proposition, the number of primes in a product must be allowed to be 1, since a prime number itself is a product of one prime. If n is a positive integer, we call an expression $n = p_1 \cdots p_k$, where $p_1, \ldots, p_k$ are prime numbers, a *prime factorization* of n. Here are some examples of prime factorizations:

$$30 = 2 \cdot 3 \cdot 5, \;\; 12 = 2 \cdot 2 \cdot 3, \;\; 13 = 13 .$$

A suitable statement to attempt to prove by induction is easy to design: for $n \geq 2$, let $P(n)$ be the statement that n is equal to a product of prime numbers.
Clearly $P(2)$ is true, as $2 = 2$ is a prime factorization of 2. However, it is not clear at all how to go about showing that $P(n) \Rightarrow P(n+1)$. In fact this cannot be done, since the primes in the prime factorization of n do not occur in the factorization of $n+1$.
However, all is not lost. We shall use the following, apparently stronger, principle of induction.

Principle of Strong Mathematical Induction
Suppose that for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things:
(a) *$P(k)$ is true;*
(b) *for all n, if $P(k), P(k+1), \ldots, P(n)$ are all true, then $P(n+1)$ is also true;*
then $P(n)$ is true for all $n \geq k$.

The logic behind this principle is not really any different from that behind the old principle: by (a), $P(k)$ is true. By (b), $P(k) \Rightarrow P(k+1)$, hence $P(k+1)$ is true. By (b) again, $P(k), P(k+1) \Rightarrow P(k+2)$, hence $P(k+2)$ is true, and so on.
(In fact, the Principle of Strong Induction is actually implied by the old principle. To see this, let $Q(n)$ be the statement that all of $P(k), \ldots, P(n)$ are true. Suppose we have proved (a) and (b) of Strong Induction. Then by (a), $Q(k)$ is true, and by (b), $Q(n) \Rightarrow Q(n+1)$. Hence by the old principle, $Q(n)$ is true for all $n \geq k$, and therefore so is $P(n)$.)
Let us now apply Strong Induction to prove Proposition 8.1.

Proof of Proposition 8.1 For $n \geq 2$, let $P(n)$ be the statement that n is equal to a product of prime numbers. As we have already remarked, $P(2)$ is true.
Now for part (b) of Strong Induction. Suppose that $P(2), \ldots, P(n)$ are all true. This means that every integer between 2 and n has a prime factorization. Now consider $n+1$. If $n+1$ is prime, then it certainly has a prime factorization (as a product of 1 prime). If $n+1$ is not prime, then by the definition of a prime number, there is an integer a dividing $n+1$ such that $a \neq 1$ or $n+1$. Writing $b = \frac{n+1}{a}$, we then have

$$n+1 = ab \quad \text{and} \quad a, b \in \{2, 3, \ldots, n\} .$$

By assumption, $P(a)$ and $P(b)$ are both true, i.e., a and b have prime factorizations. Say

$$a = p_1 \cdots p_k, \;\; b = q_1 \cdots q_l ,$$

where all the p_i and q_i are prime numbers. Then

$$n+1 = ab = p_1 \cdots p_k q_1 \cdots q_l .$$

This is an expression for $n+1$ as a product of prime numbers.
We have now shown that $P(2), \ldots, P(n) \Rightarrow P(n+1)$. Therefore, $P(n)$ is true for all $n \geq 2$, by Strong Induction.

Example 8.10
Suppose we are given a sequence of integers $u_0, u_1, u_2, \ldots, u_n, \ldots$ such that $u_0 = 2, u_1 = 3$ and

$$u_{n+1} = 3u_n - 2u_{n-1}$$

for all $n \geq 1$. (Such an equation is called a *recurrence relation* for the sequence.) Can we find a formula for u_n?
Using the equation with $n = 1$, we get $u_2 = 3u_1 - 2u_0 = 5$, and likewise $u_3 = 9, u_4 = 17, u_5 = 33, u_6 = 65$. Is there an obvious pattern? Yes, a reasonable guess seems to be that $u_n = 2^n + 1$.
So let us try to prove by Strong Induction that $u_n = 2^n + 1$. If this is the statement $P(n)$, then $P(0)$ is true, as $u_0 = 2^0 + 1 = 2$. Suppose $P(0), P(1), \ldots, P(n)$ are all true. Then $u_n = 2^n + 1$ and $u_{n-1} = 2^{n-1} + 1$. Hence from the recurrence relation,

$$u_{n+1} = 3(2^n + 1) - 2\left(2^{n-1} + 1\right) = 3 \cdot 2^n - 2^n + 1 = 2^{n+1} + 1 ,$$

which shows $P(n+1)$ is true. Therefore, $u_n = 2^n + 1$ for all n, by Strong Induction.

Exercises for Chapter 8

1. Prove by induction that it is possible to pay, without requiring change, any whole number of roubles greater than 7, with banknotes of value 3 roubles and 5 roubles.