7. Let $n \geq 2$ be an integer. Prove that n is prime if and only if for every integer a, either $\text{hcf}(a,n) = 1$ or $n|a$.

8. Let a,b be coprime positive integers. Prove that for any integer n there exist integers s,t with $s > 0$ such that $sa + tb = n$.

9. After a particularly exciting viewing of the new Danish thriller *Den hele tal*, critic Ivor Smallbrain repairs for refreshment to the prison's fast-food outlet O'Ducks. He decides that he'd like to eat some delicious Chicken O'Nuggets. These are sold in packs of two sizes — one containing 4 O'Nuggets, and the other containing 9 O'Nuggets.

   Prove that for any integer $n > 23$, it is possible for Ivor to buy n O'Nuggets (assuming he has enough money).

   Perversely however, Ivor decides that he must buy exactly 23 O'Nuggets, no more and no less. Is he able to do this ?

   Generalize this question, replacing 4 and 9 by any pair a,b of coprime positive integers: find an integer N (depending on a and b), such that for any integer $n > N$ it is possible to find integers $s,t \geq 0$ satisfying $sa + tb = n$, but no such s,t exist satisfying $sa + tb = N$.

Chapter 12

Prime Factorization

We have already seen in Chapter 8 (Proposition 8.1) that every integer greater then 1 is equal to a product of prime numbers, i.e., has a prime factorization. The main result of this chapter, the Fundamental Theorem of Arithmetic, tells us that this prime factorization is unique — in other words, there is essentially only one way of writing an integer as a product of primes. (In case you think this is somehow "obvious," have a look at Exercise 6 at the end of the chapter, to find an example of a number system where prime factorization is *not* unique.)

The Fundamental Theorem of Arithmetic may not seem terribly thrilling to you at first sight. However, it is in fact one of the most important properties of the integers, and has many consequences. I will endeavor to thrill you a little by giving a few such consequences after we have proved the theorem.

The Fundamental Theorem of Arithmetic

Without further ado then, let us state and prove the theorem.

THEOREM 12.1 (Fundamental Theorem of Arithmetic)
Let n be an integer with $n \geq 2$.

(I) Then n is equal to a product of prime numbers: we have

$$n = p_1 \cdots p_k$$

where $p_1, \ldots, p_k$ are primes and $p_1 \leq p_2 \leq \ldots \leq p_k$.

(II) This prime factorization of n is unique: in other words, if

$$n = p_1 \cdots p_k = q_1 \cdots q_l$$