edges each vertex belongs to. We record these numbers for the Platonic solids:

	V	E	F	n	r
tetrahedron	4	6	4	3	3
cube	8	12	6	4	3
octahedron	6	12	8	3	4
icosahedron	12	30	20	3	5
dodecahedron	20	30	12	5	3

As you might have guessed from the name, the Platonic solids were known to the Greeks. They are the most symmetrical, elegant and robust of solids, so it is natural to look for more regular polyhedra. Remarkably, though perhaps disappointingly, there are no others. This fact is another theorem of the great Euler. The proof is a wonderful application of Euler's formula 9.1. Here it is.

THEOREM 9.3

The only regular convex polyhedra are the five Platonic solids.

PROOF Suppose we have a regular polyhedron with parameters V, E, F, n and r as defined above.

First we need to show some relationships between these parameters. We shall prove first that

$$2E = nF \,. \tag{9.1}$$

To prove this, let us calculate the number of pairs

$$e, f$$

where e is an edge, f is a face, and e lies on f. Well, there are E possibilities for the edge e, and each lies in 2 faces f; so the number of such pairs e, f is equal to $2E$. On the other hand, there are F possibilities for the face f, and each has n edges e; so the number of such pairs e, f is also equal to nF. Therefore, $2E = nF$, proving (9.1).

Next we show that

$$2E = rV \,. \tag{9.2}$$

The proof of this is quite similar: count the pairs

$$v, e$$

where v is a vertex, e an edge, and v lies on e. There are E edges e, and each has 2 vertices v, so the number of such pairs v, e is $2E$; on the other hand, there are V vertices v, and each lies on r edges; so the number of such pairs is also rV. This proves (9.2).

At this point we appeal to Euler's formula 9.1:

$$V - E + F = 2 \,.$$

Substituting $V = \frac{2E}{r}, F = \frac{2E}{n}$ from (9.1) and (9.2), we obtain $\frac{2E}{r} - E + \frac{2E}{n} = 2$, hence

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \,. \tag{9.3}$$

Now we know that $n \geq 3$, as a polygon must have at least 3 sides; likewise $r \geq 3$, since it is geometrically clear that in a polyhedron a vertex must belong to at least 3 edges. By (9.3), it certainly cannot be the case that both $n \geq 4$ and $r \geq 4$, since this would make the left-hand side of (9.3) at most $\frac{1}{2}$, whereas the right-hand side is more than $\frac{1}{2}$. It follows that either $n = 3$ or $r = 3$.

If $n = 3$, then (9.3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E} \,.$$

The right-hand side is greater than $\frac{1}{6}$, and hence $r < 6$. Therefore, $r = 3$, 4 or 5 and $E = 6$, 12 or 30, respectively. The possible values of V, F are given by (9.1) and (9.2).

Likewise, if $r = 3$, (9.3) becomes $\frac{1}{n} = \frac{1}{6} + \frac{1}{E}$, and we argue similarly that $n = 3, 4$ or 5 and $E = 6, 12$ or 30, respectively.

We have now shown that the numbers V, E, F, n, r for a regular polyhedron must be one of the possibilities in the following table:

V	E	F	n	r
4	6	4	3	3
8	12	6	4	3
6	12	8	3	4
12	30	20	3	5
20	30	12	5	3

These are the parameter sets of the tetrahedron, cube, octahedron, icosahedron and dodecahedron, respectively. To complete the proof we now only have to show that each Platonic solid is the only regular solid with its particular parameter set. This is a simple geometric argument, and we present it just for the last parameter set — the proofs for the other sets are entirely similar.

So suppose we have a regular polyhedron with 20 pentagonal faces, each vertex lying on 3 edges. Focus on a particular vertex. At this vertex there is only one way of fitting three pentagonal faces together: