At each of the other vertices of these three pentagons, there is likewise only one way of fitting two further pentagons together. Carrying on this argument with all new vertices, we see that there is at most one way to make a regular solid with these parameters. Since the dodecahedron is such a solid, it is the only one. This completes the proof. ∎

Exercises for Chapter 9

1. Consider a convex polyhedron, all of whose faces are squares or regular pentagons. Say there are m squares and n pentagons. Assume that each vertex lies on exactly 3 edges.

   (a) Show that for this polyhedron, the following equations hold:

   $$3V = 2E, \ 4m + 5n = 2E, \ m + n = F.$$

   (b) Using Euler's formula, deduce that $2m + n = 12$.

   (c) Find examples of such polyhedra for as many different values of m as you can.

2. Prove that for a convex polyhedron with V vertices, E edges and F faces, the following inequalities are true:

   $$2E \geq 3F, \quad \text{and} \quad 2E \geq 3V.$$

   Deduce using Euler's formula that

   $$2V \geq F + 4, \ 3V \geq E + 6, \ 2F \geq V + 4 \ \text{ and } \ 3F \geq E + 6.$$

   Give an example of a convex polyhedron for which all these inequalities are equalities (i.e., $2V = F + 4$, etc.).

3. Prove that if a connected plane graph has v vertices and e edges, then $e \leq 3v - 6$.

4. Prove that it is impossible to make a football out of exactly 9 squares and m octagons, where $m \geq 4$. (In this context, a "football" is a convex polyhedron in which at least 3 edges meet at each vertex.)

5. Draw all the connected plane graphs with 4 edges, and all the connected plane graphs with 4 vertices.

6. Let K_n denote the graph with n vertices in which any two vertices are joined by an edge. So for example K_2 consists of 2 vertices joined by an edge, and K_3 is a triangle.

   Prove that it is possible to draw K_4 as a plane graph. (Note that edges do not have to be drawn as straight lines.)

   Prove that it is impossible to draw K_5 as a plane graph. (*Hint:* Use the inequality in Question 3 cleverly.)

7. Prove that every connected plane graph has a vertex that is joined to at most five other vertices. (*Hint:* Assume every vertex is joined to at least 6 others, and try to use Question 3 to get a contradiction.)

8. Critic Ivor Smallbrain has been thrown into prison for libelling the great film director Michael Loser. During one of his needlework classes in prison, Ivor is given a pile of pieces of leather in the shapes of regular pentagons and regular hexagons, and is told to sew some of these together into a convex polyhedron (which will then be used as a football). He is told that each vertex must lie on exactly 3 edges. Ivor immediately exclaims, "Then I need exactly 12 pentagonal pieces!"

   Prove that Ivor is correct.