equations are very hard, or impossible, to solve — for instance, even the equation $x^2 = y^3 + k$ has not been completely solved for all values of k. However, I have chosen a nice example, in that Equation (12.1) can be solved fairly easily (as you will see), but the solution is not totally trivial and involves use of the consequence 12.4 of the Fundamental Theorem 12.1.

Let us then go about solving Equation (12.1) for $x, y \in \mathbb{Z}$. First we rewrite it as $y^3 = 4x^2 - 1$, and then cleverly factorize the right-hand side to get

$$y^3 = (2x+1)(2x-1) .$$

The factors $2x+1, 2x-1$ are both odd integers, and their highest common factor divides their difference, which is 2. Hence

$$\mathrm{hcf}(2x+1, 2x-1) = 1 .$$

Thus, $2x+1$ and $2x-1$ are coprime to each other, and their product is y^3, a cube. By Proposition 12.4(b), it follows that $2x+1$ and $2x-1$ are themselves both cubes. However, from the list of cubes $\ldots, -8, -1, 0, 1, 8, 27, \ldots$ it is apparent that the only two cubes that differ by 2 are $1, -1$. Therefore, $x = 0$ and we have shown that the only even square that exceeds a cube by 1 is 0. In other words, there are no non-zero such squares.

Exercises for Chapter 12

1. Find the prime factorization of 111111.

2. (a) Which positive integers have exactly three positive divisors?

   (b) Which positive integers have exactly four positive divisors?

   (c) Suppose $n \geq 2$ is an integer with the property that whenever a prime p divides n, p^2 also divides n (i.e. all primes in the prime factorization of n appear at least to the power 2). Prove that n can be written as the product of a square and a cube.

3. Suppose that n is a positive integer such that $p = 2^n - 1$ is prime. (The first few such primes are $3, 7, 31, \ldots$.) Define

$$N = 2^{n-1}p .$$

   List all positive integers that divide N. Prove that the sum of all these divisors, including 1 but not N itself, is equal to N.

   A positive integer that is equal to the sum of all its divisors (including 1 but not itself) is called a *perfect* number. Write down four perfect numbers.

4. Prove that $lcm(a,b) = ab/hcf(a,b)$ for any positive integers a, b without using prime factorization.

5. (a) Prove that $2^{\frac{1}{3}}$ and $3^{\frac{1}{3}}$ are irrational.

   (b) Let m and n be positive integers. Prove that $m^{\frac{1}{n}}$ is rational if and only if m is an n^{th} power (i.e., $m = c^n$ for some integer c).

6. Let E be the set of all positive even integers. We call a number e in E "prima" if e cannot be expressed as a product of two other members of E.

   (i) Show that 6 is prima but 4 is not.

   (ii) What is the general form of a prima in E?

   (iii) Prove that every element of E is equal to a product of primas.

   (iv) Give an example to show that E does not satisfy a "unique prima factorization theorem" (i.e., find an element of E that has two different factorizations as a product of primas).

7. (a) Which pairs of positive integers m, n have $hcf(m,n) = 50$ and $lcm(m,n) = 1500$?

   (b) Show that if m, n are positive integers, then $hcf(m,n)$ divides $lcm(m,n)$. When does $hcf(m,n) = lcm(m,n)$?

   (c) Show that if m, n are positive integers, then there are coprime integers x, y such that x divides m, y divides n, and $xy = lcm(m,n)$.

8. Find all solutions $x, y \in \mathbb{Z}$ to the following Diophantine equations:

   (a) $x^2 = y^3$

   (b) $x^2 - x = y^3$

   (c) $x^2 = y^4 - 77$

   (d) $x^3 = 4y^2 + 4y - 3$.

9. Languishing in his prison cell, critic Ivor Smallbrain is dreaming. In his dream he is on the Pacific island of Nefertiti, eating coconuts on a beach by a calm blue lagoon. Suddenly the king of Nefertiti approaches him, saying, "Your head will be chopped off unless you answer this riddle: Is it possible for the sixth power of an integer to exceed the fifth power of another integer by 16?" Feverishly, Ivor writes some calculations in the sand, and eventually answers, "Oh Great King, no it is not possible." The