Example 10.1

(1) Let

$$S=\left\{\frac{1}{n}\,\middle|\, n\in\mathbb{N}\right\}=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\right\}.$$

Then 1 is an upper bound for S; so are 2, 17, and indeed any number that is at least 1. Also 0 is a lower bound for S, and so is any number less than or equal to 0.

(2) If $S=\mathbb{Z}$, then S has no upper or lower bound.

(3) If

$$S=\left\{x \mid x\in\mathbb{Q}, x^2<2\right\},$$

i.e., the set of rationals with square less than 2, then $\sqrt{2}$ is an upper bound for S, and $-\sqrt{2}$ is a lower bound.

As we see from these examples, a set can have many upper bounds. It turns out to be a fundamental question to ask whether, among all the upper bounds, there is always a least one. Let us first formally define such a thing.

DEFINITION *Let S be a non-empty subset of $\mathbb{R}$, and suppose S has an upper bound. We say that a real number c is a* least upper bound *(abbreviated, LUB), if the following two conditions hold:*

(i) c is an upper bound for S, and

(ii) if u is any other upper bound for S, then $u\geq c$.

Similarly, d is a greatest lower bound *(GLB) for S if*

(a) d is a lower bound for S, and

(b) if l is any other lower bound for S, then $l\leq d$.

Example 10.2

Let $S=\{\frac{1}{n}\mid n\in\mathbb{N}\}$ as in Example 10.1(1) above.

We claim that 1 is a LUB for S. To see this, observe that 1 is an upper bound; and any other upper bound is at least 1, since $1\in S$.

We also claim that 0 is a GLB for S. This is not quite so obvious. First, 0 is a lower bound. Let l be another lower bound for S. If $l>0$, then we can find $n\in\mathbb{N}$ such that $\frac{1}{n}<l$; but $\frac{1}{n}\in S$, so this is not possible as l is a lower bound for S. Hence $l\leq 0$, which proves that 0 is a GLB for S.

The next result is absolutely fundamental to the study of the real numbers.

THEOREM 10.1

Let S be a non-empty subset of $\mathbb{R}$.

(I) If S has a lower bound, then it has a greatest lower bound.

(II) If S has an upper bound, then it has a least upper bound.

PROOF (I) Suppose S has a lower bound. Now every member s of S has a decimal expression

$$s=s_0\cdot s_1s_2s_3\dots$$

Since S has a lower bound, the integers s_0 occurring cannot decrease indefinitely; so there must be a smallest integer s_0 occurring; call it a_0.

Among all the members of S with decimal expressions starting with this integer a_0 (i.e., starting $a_0\cdot s_1s_2\dots$), choose one with the digit s_1 in the first decimal place as small as possible; call this digit a_1.

Similarly, among all the members of S starting $a_0\cdot a_1s_2s_3\dots$, choose one with the digit in the second decimal place as small as possible; call this digit a_2. Carry on choosing digits $a_3,a_4,\dots$ in this way. Now define the real number

$$d=a_0\cdot a_1a_2a_3\dots$$

We shall show that d is a greatest lower bound for S.

First we must show that d is a lower bound. Let $s\in S$, with $s=s_0\cdot s_1s_2s_3\dots$. If $s\neq d$, let the first decimal place where s and d disagree be the k^{th} place. Then $s=a_0\cdot a_1\dots a_{k-1}s_k\dots$ with $s_k\neq a_k$ (possibly $k=0$ of course). By our choice of a_k (as the least k^{th} digit occurring among members of S that start $a_0\cdot a_1\dots a_{k-1}\dots$), we must have $s_k>a_k$. This means that $s\geq d$. Hence, d is a lower bound for S.

Now we show d is a GLB. Let l be a lower bound for S, and let $l=l_0\cdot l_1l_2\dots$. We need to prove that $d\geq l$. If $d\neq l$, let the first decimal place where d and l disagree be the r^{th} place. Then $l=a_0\cdot a_1\dots a_{r-1}l_r\dots$ with $l_r\neq a_r$. From our choice of the digits $a_0,a_1,\dots$, we know that there is a member s of S which starts $s=a_0\cdot a_1\dots a_{r-1}a_r\dots$. Since l is a lower bound for S, we have $s\geq l$ and hence $a_r>l_r$. Therefore $d\geq l$, proving that d is a GLB for S.

This completes the proof of (I).

(II) We shall use a simple trick to deduce (II) from (I). Suppose S has an upper bound, say u. Define the set $-S$ to consist of the negatives of all members of S; that is,

$$-S=\{-s \mid s\in S\}.$$

As $s\leq u$ for all $s\in S$, we have $-s\geq -u$ for all $-s\in -S$, and so $-u$ is a lower bound for $-S$. Therefore by (I), the set $-S$ has a GLB; call it c.