Chapter 10

Introduction to Analysis

We saw in Chapter 2 how to prove the existence of the real number $\sqrt{2}$, and more generally of $\sqrt{n}$ for any positive integer n, by means of a clever geometrical construction. However, proving the existence of cube roots, and more generally, a real n^{th} root $x^{\frac{1}{n}}$, for any positive real number x is much harder, and relies on a much deeper study of the reals. We shall undertake such a study in this chapter. Our main focus will be the proof of the existence of n^{th} roots. (We have already stated this result in anticipation as Proposition 5.1, and used it in later chapters.) However, I should point out that the material introduced in this chapter is a fundamental starting point for a huge area of mathematics called Analysis, which is basically the study of functions of real and complex numbers.

Upper and Lower Bounds

Our analysis will be based on the theory of *bounds* for sets of real numbers. Here is the definition.

DEFINITION *Let S be a non-empty subset of $\mathbb{R}$. (So S is a set consisting of some real numbers, and $S \neq \emptyset$). We say that a real number u is an* upper bound *for S if*

$$s \leq u \text{ for all } s \in S .$$

Likewise, l is a lower bound *for S if*

$$s \geq l \text{ for all } s \in S .$$