We can carry on drawing lines and counting the regions they form, which leads us naturally to a general question:

If we draw n straight lines in the plane, no three going through the same point, and no two parallel, how many regions do they divide the plane into?

The conditions about not going through the same point and not being parallel may seem strange, but in fact they are very natural: if you draw lines at random, it is very unlikely that two will be parallel or that three will pass through the same point — so you could say the lines in the question are "random" lines. Technically, they are said to be ***lines in general position***.

The answers to the question for $n = 1, 2, 3, 4$ are $2, 4, 7, 11$. Even from this flimsy evidence you have probably spotted a pattern — the difference between successive terms seems to be increasing by 1 each time. Can we predict a formula from this pattern? Yes, of course we can: the number of regions for one line is two, for two lines is $2+2$, for three lines is $2+2+3$, for four lines is $2+2+3+4$, so we predict that the number of regions for n lines is

$$2+2+3+4+\cdots+n .$$

This is just $1+\sum_{r=1}^{n} r$, which by Example 8.6 is equal to $1+\frac{1}{2}n(n+1)$.

Let us therefore attempt to prove the following statement $P(n)$ by induction: the number of regions formed in the plane by n straight lines in general position is $\frac{1}{2}(n^2+n+2)$.

First $P(1)$ is true, as the number of regions for one line is $2=\frac{1}{2}(1^2+1+2)$.

Now suppose $P(n)$ is true, so n lines in general position form $\frac{1}{2}(n^2+n+2)$ regions. Draw in an $(n+1)^{th}$ line. Since it is not parallel to any of the others, this line meets each of the other n lines in a point, and these n points of intersection divide the $(n+1)^{th}$ line into $n+1$ pieces. Each of these pieces divides an old region into two new ones. Hence, when the $(n+1)^{th}$ line is drawn, the number of regions increases by $n+1$. (If this argument is not clear to you, try drawing a picture to illustrate it when $n=3$ or 4.) Consequently the number of regions with $n+1$ lines is equal to $\frac{1}{2}(n^2+n+2)+n+1$. Check that this is equal to $\frac{1}{2}((n+1)^2+(n+1)+2)$.

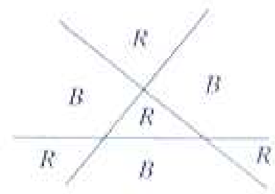
We have now shown that $P(n) \Rightarrow P(n+1)$. Hence, by induction, $P(n)$ is true for all $n \geq 1$.

Induction is a much more powerful method than you might think. It can often be used to prove statements that do not actually explicitly mention an integer n. In such instances, one must imaginatively design a suitable statement $P(n)$ to fit in with the problem, and then try to prove $P(n)$ by induction. In the next two examples this is fairly easy to do. The next chapter, however, will be devoted to an example of a proof by induction where the statement $P(n)$ lies a long way away from the initial problem.

Example 8.9

Some straight lines are drawn in the plane, forming regions as in the previous example. Show that it is possible to colour each region either red or blue, in such a way that no two neighbouring regions have the same colour.

For example, here is such a colouring when there are three lines:



How do we design a suitable statement $P(n)$ for this problem? This is very simple: just take $P(n)$ to be the statement that the regions formed by n straight lines and the plane can be coloured in the required way.

Actually, the proof of $P(n)$ by induction is so neat and elegant that I would hate to deprive you of the pleasure of thinking about it, so I leave it to you. (It is set as Exercise 14 at the end of the chapter in case you forget.)

Prime Factorization

In the next example, we prove a very important result about the integers. First we need a definition:

DEFINITION A prime number *is a positive integer p such that $p \geq 2$ and the only positive integers dividing p are 1 and p.*

You are probably familiar to some extent with prime numbers. The first few are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$.

The important result we shall prove is the following: