Thus, Example 8.5 says that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1},$$

and Example 8.1 says

$$\sum_{r=1}^{n} (2r-1) = n^2 .$$

Notice that if a, b, c are constants, then

$$\sum_{r=1}^{n} (af_r + bg_r + c) = a\sum_{r=1}^{n} f_r + b\sum_{r=1}^{n} g_r + cn , \qquad (*)$$

since the left-hand side is equal to

$$\begin{aligned}&(af_1 + bg_1 + c) + \cdots + (af_n + bg_n + c)\\ &\quad = a(f_1 + \cdots + f_n) + b(g_1 + \cdots + g_n) + (c + \cdots + c) ,\end{aligned}$$

which is the right-hand side.

The Equation $(*)$ is quite useful for manipulating sums. Here is an elementary example using it.

Example 8.6

Find a formula for $\sum_{r=1}^{n} r \ (= 1+2+\cdots+n)$.

Answer Write $s_n = \sum_{r=1}^{n} r$. By Example 8.1, $\sum_{r=1}^{n} (2r-1) = n^2$, so using Equation $(*)$,

$$n^2 = \sum_{r=1}^{n} (2r-1) = 2\sum_{r=1}^{n} r - n = 2s_n - n .$$

Hence, $s_n = \frac{1}{2}n(n+1)$.

So we know the sum of the first n positive integers. What about the sum of the first n squares?

Example 8.7

Find a formula for $\sum_{r=1}^{n} r^2 \ (= 1^2 + 2^2 + \cdots + n^2)$.

Answer We first try to guess the answer (intelligently). The first few values $n = 1,2,3,4$ give sums $1, 5, 14, 30$. It is not easy to guess a formula from these

values, so yet a smidgeon more intelligence is required. The sum we are trying to find is the sum of n terms of a quadratic nature, so it seems reasonable to look for a formula for the sum which is a cubic in n, say $an^3 + bn^2 + cn + d$.

What should a, b, c, d be? Well, they have to fit in with the values of the sum for $n = 1,2,3,4$, and hence must satisfy the following equations:

$$n = 1 : 1 = a + b + c + d \qquad (8.1)$$
$$n = 2 : 5 = 8a + 4b + 2c + d \qquad (8.2)$$
$$n = 3 : 14 = 27a + 9b + 3c + d \qquad (8.3)$$
$$n = 4 : 30 = 64a + 16b + 4c + d \qquad (8.4)$$

Equations (8.2)-(8.1), (8.3)-(8.2), (8.4)-(8.3) then give $4 = 7a + 3b + c$, $9 = 19a + 5b + c$, $16 = 37a + 7b + c$. Subtraction of these gives $5 = 12a + 2b$, $7 = 18a + 2b$. Hence we get the solution

$$a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}, d = 0 .$$

Hence, our (intelligent) guess is that

$$\sum_{r=1}^{n} r^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1) .$$

This turns out to be correct, and we leave it to the reader to prove it by induction. (It is set as Exercise 2 at the end of the chapter in case you forget.)

(Actually, there is a much better way of working out a formula for $\sum_{r=1}^{n} r^2$, given in Exercise 4 at the end of the chapter.)

The next example is a nice geometric proof by induction.

Example 8.8

Lines in the plane. If we draw a straight line in the plane, it divides the plane into two regions. If we draw another, not parallel to the first, the two lines divide the plane into four regions. Likewise, three lines, not all going through the same point, and no two of which are parallel, divide the plane into seven regions: