It is now very easy to see that $-c$ is a LUB for S: first, $-s \geq c$ for all $-s \in -S$, hence $s \leq -c$ for all $s \in S$, and so $-c$ is an upper bound for S. If u is any other upper bound, then $-u$ is a lower bound for $-S$, so $-u \leq c$, which implies $u \geq -c$. Hence, $-c$ is a LUB for S, as required. ∎

Note If a set S has a LUB, it is easy to see that it has only one LUB. We leave this to the reader (it is Exercise 3 at the end of the chapter in case you forget). So it makes sense to talk about *the* least upper bound of S. We sometimes denote this by LUB(S). Likewise, a set S with a lower bound has only one GLB, denoted by GLB(S).

As we have said, Theorem 10.1 underlies the whole of the theory of the real numbers, and you will see it used many times in your future study of mathematics. For now, we give one application by proving the existence of n^{th} roots.

Existence of n^{th} Roots

We aim now to prove the following result, already stated as Proposition 5.1.

PROPOSITION 10.1

Let n be a positive integer. If x is a positive real number, then there is exactly one positive real number y such that $y^n = x$.

The idea behind our proof of this result is very simple, but the proof itself involves quite a bit of notation, which may slightly obscure the idea at first sight. So to make everything crystal clear, we first present an example to illustrate the idea.

Example 10.3

In this example, we prove the existence of a real number c such that $c^3 = 2$. That is, we prove the existence of $2^{\frac{1}{3}}$, the real cube root of 2.

The key idea is to define the following set of real numbers:

$$S = \{x \mid x \in \mathbb{R}, x^3 < 2\}\ .$$

Thus, S is the set of all real numbers whose cube is less than 2.

First note that S has an upper bound: for example, 2 is an upper bound, since if $x^3 < 2$ then $x < 2$. Therefore, by Theorem 10.1, S has a least upper bound. Say

$$c = \mathrm{LUB}(S)\ .$$

We shall show that $c^3 = 2$. We do this by contradiction.

Assume then that $c^3 \neq 2$. Then either $c^3 < 2$ or $c^3 > 2$. We consider these two possibilities separately, in each case obtaining a contradiction.

Case 1 Assume that $c^3 < 2$. Our strategy in this case is to find a small number $\alpha > 0$ such that $(c+\alpha)^3 < 2$ still; this will mean that $c + \alpha \in S$, whereas c is an upper bound for S, a contradiction.

To find α, we argue as follows. (We have chosen to present the steps "in reverse" in order to make it clear how the argument was found.) We have

$$\begin{aligned}(c+\alpha)^3 < 2 &\Leftarrow c^3 + 3c^2\alpha + 3c\alpha^2 + \alpha^3 < 2\\ &\Leftarrow 2 - c^3 > 3c^2\alpha + 3c\alpha^2 + \alpha^3\\ &\Leftarrow 2 - c^3 > \alpha(3c^2 + 3c + 1)\ \text{ and }\ 0 < \alpha < 1\end{aligned}$$

(The last inequality follows since when $0 < \alpha < 1$, we have $\alpha^2 < \alpha$ and $\alpha^3 < \alpha$.) Since $2 - c^3 > 0$ we can choose α such that

$$0 < \alpha < 1 \ \text{ and }\ \alpha < \frac{2 - c^3}{3c^2 + 3c + 1}\ .$$

Then by the above implications it follows that $(c+\alpha)^3 < 2$. This leads to a contradiction, as explained before.

Case 2 Now assume that $c^3 > 2$. In this case our strategy is to find a small number $\beta > 0$ such that $(c-\beta)^3 > 2$ still. If we do this, then for $x \in S$ we have $x^3 < 2 < (c-\beta)^3$, hence $x < c - \beta$, and so $c - \beta$ is an upper bound for S. However c is the LUB of S, so this is a contradiction.

To find β, note that

$$\begin{aligned}(c-\beta)^3 > 2 &\Leftarrow c^3 - 2 > 3c^2\beta - 3c\beta^2 + \beta^3\\ &\Leftarrow c^3 - 2 > \beta(3c^2 + 3c + 1)\ \text{ and }\ 0 < \beta < 1\ .\end{aligned}$$

Since $c^3 - 2 > 0$ we can choose β such that

$$0 < \beta < 1 \ \text{ and }\ \beta < \frac{c^3 - 2}{3c^2 + 3c + 1}\ .$$

Then by the above implications it follows that $(c-\beta)^3 > 2$. This leads to a contradiction, as explained before.

Thus we have reached a contradiction in both Cases 1 and 2, from which we conclude that $c^3 = 2$. In other words, c is the real cube root of 2.