The next example also uses the slightly modified Principle of Mathematical Induction II. In it, for a positive integer n we define

$$n! = n(n-1)(n-2)\ldots 3\cdot 2\cdot 1\,,$$

the product of all the integers between 1 and n. The symbol $n!$ is usually referred to as *n factorial*. By convention, we also define $0! = 1$.

Example 8.4
For which positive integers n is $2^n < n!$?

Answer Let $P(n)$ be the statement that $2^n < n!$. Observe that

$$2^1 > 1!,\ 2^2 > 2!,\ 2^3 > 3!,\ 2^4 < 4!,\ 2^5 < 5!\,,$$

so $P(1), P(2), P(3)$ are false, while $P(4), P(5)$ are true. Therefore, it seems sensible to try to prove $P(n)$ is true for all $n \geq 4$.
First, $P(4)$ is true, as observed above.
Now suppose n is an integer at least 4, and $P(n)$ is true. Thus

$$2^n < n!$$

Multiplying both sides by 2, we get

$$2^{n+1} < 2(n!)\,.$$

Since $2 < n+1$, $2(n!) < (n+1)n! = (n+1)!$, and hence $2^{n+1} < (n+1)!$. This shows that $P(n) \Rightarrow P(n+1)$. Therefore, by induction, $P(n)$ is true for all $n \geq 4$.

Guessing the Answer

Some problems cannot immediately be tackled using induction, but first require some intelligent guesswork. Here is an example.

Example 8.5
Find a formula for the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}\,.$$

Answer Calculate this sum for the first few values of n:

$$n = 1: \frac{1}{1\cdot 2} = \frac{1}{2}\,,$$

$$n = 2: \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}\,,$$
$$n = 3: \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}\,.$$

We intelligently spot a pattern in these answers, and guess that the sum of n terms is probably $\frac{n}{n+1}$. Hence we let $P(n)$ be the statement

$$P(n): \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

and attempt to prove $P(n)$ true for all $n \geq 1$ by induction.
First, $P(1)$ is true, as noted above.
Now assume $P(n)$ is true, so

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}\,.$$

Adding $\frac{1}{(n+1)(n+2)}$ to both sides gives

$$\frac{1}{1\cdot 2} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
$$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}\,.$$

Hence $P(n) \Rightarrow P(n+1)$. So by induction $P(n)$ is true for all $n \geq 1$.

The Σ Notation

Before proceeding with the next example, we introduce an important notation for writing down sums of many terms. If $f_1, f_2, \ldots, f_n$ are numbers, we abbreviate the sum of all of them by

$$f_1 + f_2 + \cdots + f_n = \sum_{r=1}^{n} f_r\,.$$

(The symbol Σ is the Greek capital letter "Sigma," so this is often called the "Sigma notation.") For example, setting $f_r = \frac{1}{r(r+1)}$, we have

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \sum_{r=1}^{n} \frac{1}{r(r+1)}\,.$$