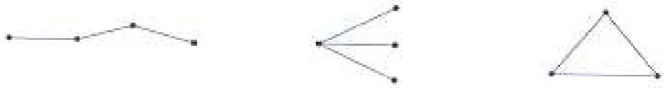
Proof of Theorem 9.2 Here is the statement $P(n)$ that we are going to try to prove by induction:

$P(n)$: every connected plane graph with n edges satisfies the formula $v-n+f=1$.

Notice that $P(n)$ is a statement about lots of plane graphs. $P(1)$ says that every connected plane graph with 1 edge satisfies the formula; there is only one such graph:

This has 2 vertices, 1 edge and 0 faces, and since $2-1+0=1$, it satisfies the formula. Likewise, $P(2)$ says that the graph

satisfies the formula, which it does, as $v=3, e=2, f=0$. For $P(3)$, there are three different connected plane graphs with 3 edges:



Each satisfies the formula.

Let us prove $P(n)$ by induction. First, $P(1)$ is true, as observed in the previous paragraph.

Now assume $P(n)$ is true — so every connected plane graph with n edges satisfies the formula. We need to deduce $P(n+1)$. So consider a connected plane graph G with $n+1$ edges. Say G has v vertices and f faces. We want to prove that G satisfies the formula $v-(n+1)+f=1$.

Our strategy will be to remove a carefully chosen edge from G, so as to leave a connected plane graph with only n edges, and then use $P(n)$.

If G has at least 1 face (i.e., $f \geq 1$), we remove one edge of this face. The remaining graph G' is still connected, and has n edges, v vertices and $f-1$ faces. Since we are assuming $P(n)$, we know that G' satisfies the formula, hence

$$v-n+(f-1)=1\,.$$

Therefore $v-(n+1)+f=1$, as required.

If G has no faces at all (i.e., $f=0$), then it has at least one "end-vertex," i.e., a vertex that is joined by an edge to only one other vertex. Removing this end-vertex and its edge from G leaves a connected plane graph G'' with $v-1$ vertices, n edges and 0 faces. By $P(n)$, G'' satisfies the formula, so

$$(v-1)-n+0=1\,.$$

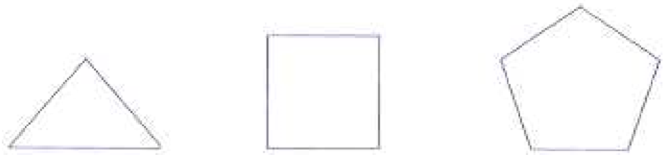
Hence $v-(n+1)+0=1$, which is the formula for G.

We have established that $P(n) \Rightarrow P(n+1)$, so $P(n)$ is true for all n by induction.

This completes the proof of Theorem 9.2, and hence of Euler's theorem 9.1.

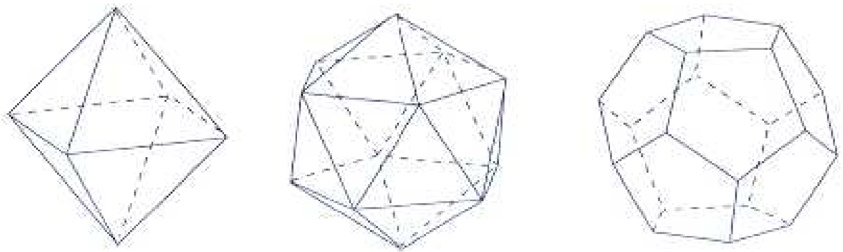
Regular and Platonic Solids

A polygon is said to be *regular* if all its sides are of equal length and all its internal angles are equal. We call a regular polygon with n sides a *regular n-gon*. Some of these shapes are probably quite familiar; for example, a regular n-gon with $n=3$ is just an equilateral triangle, $n=4$ is a square, $n=5$ is a regular pentagon, and so on:



A polyhedron is *regular* if its faces are regular polygons, all with the same number of sides, and also each vertex belongs to the same number of edges.

Three examples of regular polyhedra come more or less readily to mind: the cube, the tetrahedron and the octahedron. These are three of the famous five *Platonic solids*; the other two are the less obvious *icosahedron*, which has 20 triangular faces, and *dodecahedron*, which has 12 pentagonal faces. Here are pictures of the octahedron, icosahedron and dodecahedron:



Every regular polyhedron carries five associated numbers: three are V, E, F, and the other two are n, the number of sides on a face, and r, the number of