2. Prove by induction that $\Sigma_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.
Deduce formulae for

$$1\cdot 1+2\cdot 3+3\cdot 5+4\cdot 7+\cdots+n(2n-1) \text{ and } 1^2+3^2+5^2+\cdots(2n-1)^2.$$

3. (a) Work out $1, 1+8, 1+8+27$ and $1+8+27+64$. Guess a formula for $\Sigma_{r=1}^{n} r^3$ and prove it.

(b) Check that $1 = 0+1$, $2+3+4 = 1+8$ and $5+6+\cdots+9 = 8+27$. Find a general formula for which these are the first three cases. Prove your formula is correct.

4. Here is another way to work out $\Sigma_{r=1}^{n} r^2$. Observe that $(r+1)^3 - r^3 = 3r^2+3r+1$. Hence

$$\sum_{r=1}^{n}(r+1)^3 - r^3 = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n.$$

The left-hand side is equal to

$$(2^3-1^3)+(3^3-2^3)+(4^3-3^3)+\cdots$$
$$+((n+1)^3-n^3) = (n+1)^3-1.$$

Hence we can work out $\Sigma_{r=1}^{n} r^2$.

Carry out this calculation, and check that your formula agrees with that in Question 2..

Use the same method to work out formulae for $\Sigma_{r=1}^{n} r^3$ and $\Sigma_{r=1}^{n} r^4$.

5. Prove the following statements by induction:

(a) For all integers $n \geq 0$, the number $5^{2n} - 3^n$ is a multiple of 11.

(b) For any integer $n \geq 1$, the integer 2^{4n-1} ends with an 8.

(c) The sum of the cubes of three consecutive positive integers is always a multiple of 9.

(d) If $x \geq 2$ is a real number and $n \geq 1$ is an integer, then $x^n \geq nx$.

(e) If $n \geq 3$ is an integer, then $5^n > 4^n + 3^n + 2^n$.

6. The *Lucas sequence* is a sequence of integers $l_1, l_2, \ldots, l_n, \ldots$ such that $l_1 = 1, l_2 = 3$ and

$$l_{n+1} = l_n + l_{n-1}$$

for all $n \geq 1$. So the sequence starts $1, 3, 4, 7, 11, 18, \ldots$

Find the pattern for the remainders when l_n is divided by 3. (*Hint:* Consider the first 8 remainders, then the next 8, and so on; formulate a conjecture for the pattern, and prove it by induction.)

Is l_{2000} divisible by 3 ?

7. The *Fibonacci sequence* is a sequence of integers $f_1, f_2, \ldots, f_n, \ldots$ such that $f_1 = 1, f_2 = 1$ and

$$f_{n+1} = f_n + f_{n-1}$$

for all $n \geq 1$. Prove by strong induction that for all n,

$$f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

8. I just worked out $(2+\sqrt{3})^{50}$ on my computer and got the answer

39571031999226139563162735373.999999999999999999999999974728...

Why is this so close to an integer ?

(*Hint:* Try to use the idea of the previous question by constructing a suitable sequence.)

9. Prove that if $0 < q < \frac{1}{2}$, then for all $n \geq 1$,

$$(1+q)^n \leq 1 + 2^n q.$$

10. (a) Prove that for every integer $n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{7}{12}.$$

(b) Prove that for every integer $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

11. Just for this question, count 1 as a prime number. A well-known result in number theory says that for every integer $x \geq 3$, there is a prime number p such that $\frac{1}{2}x < p < x$. Using this result and strong induction, prove that every positive integer is equal to a sum of primes, all of which are different.

12. Here is a "proof" by induction that any two positive integers are equal (e.g., $5 = 10$):

First, a definition: if a and b are positive integers, define $\max(a,b)$ to be the larger of a and b if $a \neq b$, and to be a if $a = b$. (For instance, $\max(3,5) = 5$, $\max(3,3) = 3$.) Let $P(n)$ be the statement: "if a and b are positive integers such that $\max(a,b) = n$, then $a = b$." We prove $P(n)$