Exercises for Chapter 10

1. Which of the following sets S have an upper bound, and which have a lower bound? In the cases where these exist, state what the least upper bounds and greatest lower bounds are.

   (i) $S = \{-1, 3, 7, -2\}$
   (ii) $S = \{x \mid x \in \mathbb{R} \text{ and } |x-3| < |x+7|\}$
   (iii) $S = \{x \mid x \in \mathbb{R} \text{ and } x^3 - 3x < 0\}$
   (iv) $S = \{x \mid x \in \mathbb{N} \text{ and } x^2 = a^2 + b^2 \text{ for some } a, b \in \mathbb{N}\}$.

2. Write down proofs of the following statements about sets A and B of real numbers.

   (a) If x is an upper bound for A, and $x \in A$, then x is a least upper bound for A.

   (a) If $A \subseteq B$, then a lower bound for B is also a lower bound for A.

   (b) If $A \subseteq B$, and a greatest lower bound of A is x, and a greatest lower bound of B is y, then $y \leq x$.

3. Prove that if S is a set of real numbers, then S cannot have two different least upper bounds or greatest lower bounds.

4. Find the LUB and GLB of the following sets:

   (i) $\{x \mid x = 2^{-p} + 3^{-q} \text{ for some } p, q \in \mathbb{N}\}$
   (ii) $\{x \in \mathbb{R} \mid 3x^2 - 4x < 1\}$
   (iii) the set of all real numbers between 0 and 1 whose decimal expression contains no nines.

5. (a) Find a set of rationals having rational LUB.

   (b) Find a set of rationals having irrational LUB.

   (c) Find a set of irrationals having rational LUB.

6. Which of the following statements are true and which are false?

   (a) Every set of real numbers has a GLB.

   (b) For any real number r, there is a set of rationals having GLB equal to r.

   (c) Let $S \subseteq \mathbb{R}, T \subseteq \mathbb{R}$, and define $ST = \{st \mid s \in S, t \in T\}$, the set of all products of elements of S with elements of T. If c is the GLB of S, and d is the GLB of T, then cd is the GLB of ST.

   (d) If S is a set of real numbers such that $\text{GLB}(S) \notin S$, then S must be an infinite set. (An *infinite* set is one which does not consist of only a finite number of elements.)

7. Prove that the cubic equation $x^3 - x - 1 = 0$ has a real root (i.e., prove that there exists a real number c such that $c^3 - c - 1 = 0$). (*Hint:* Try to find c as the LUB of a suitable set.)

8. Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers (going on forever). For any integer $n \geq 1$, define T_n to be the set $\{x_n, x_{n+1}, \ldots\}$. (So for example $T_1 = \{x_1, x_2, x_3, \ldots\}$ and $T_2 = \{x_2, x_3, x_4, \ldots\}$.)

   Assume that T_1 has a lower bound. Deduce that for any n, the set T_n has a GLB, and call it b_n. Prove that $b_1 \leq b_2 \leq b_3 \leq \cdots$.

   For the following sequences $x_1, x_2, \ldots$, work out b_n, and also work out the LUB of the set $\{b_1, b_2, \ldots\}$ when it exists:

   (a) $x_1 = 1, x_2 = 2, x_3 = 3$, and in general $x_n = n$.

   (b) $x_1 = 1, x_2 = \frac{1}{2}, x_3 = \frac{1}{3}$, and in general $x_n = \frac{1}{n}$.

   (c) $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 2, x_5 = 1$, and so on, alternating between 1 and 2.

9. During the long hours in his prison cell, critic Ivor Smallbrain reckons he has managed to prove that if n is any integer with $n \geq 3$, then there do not exist any positive integers a, b, c satisfying the equation $a^n + b^n = c^n$. He modestly calls this result "Smallbrain's first theorem," and attempts to write down a proof in the margin of his theatre programme during a particularly tedious prison performance of the Hungarian classic *Bevezetés – hoz analízis*. He fails to do so because the margin is too small.

   Now let us define a sequence $x_1, x_2, x_3, \ldots$ by letting $x_n = 1$ if there exist positive integers a, b, c satisfying the equation $a^n + b^n = c^n$, and letting $x_n = 2$ otherwise. Assuming that Smallbrain's first theorem is true, find the numbers b_n defined in the last question, and the LUB of the set $\{b_1, b_2, \ldots\}$.