Introduction to Analysis

October 25, 2013

Week 7-8

1 Upper and Lower Bounds

Definition 1. Let S be a nonempty subset of $\mathbb{R}$, i.e., S is a set consisting of some real numbers and $S \neq \emptyset$. A real number u is called an **upper bound** for S if

$$s \leq u \quad \text{for all} \quad s \in S.$$

Likewise, a real number l is called a **lower bound** for S if

$$s \geq l \quad \text{for all} \quad s \in S.$$

Example 1. (a) The set $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ has upper bound 1. Of course, 2, 3 are also upper bounds.

(b) The set of even integers has no upper bound.

(c) $S = \left\{-\frac{1}{n} \mid n \in \mathbb{N}\right\}$ has upper bounds.

(d) $S = \{x \mid x \in \mathbb{Q}, x^2 < 3$ is the set of all rational numbers whose square is less than 3. Then $\sqrt{3}$ is an upper bound for S, and $-\sqrt{3}$ is a lower bound for S.

Definition 2. Let S be a nonempty subset of $\mathbb{R}$ and be bounded above. A real number c is called a **least upper bound** (short for LUB or **supremum**) for S if the following two conditions hold:

(i) c is an upper bound for S.

(ii) If u is any upper bound for S then $c \leq u$.

Similarly, let S be nonempty and be bounded below. A real number d is called a **greatest lower bound** (short for GLB or **infimum**) for S if

(i) d is a lower bound for S.

(ii) If l is any lower bound for S then $l \leq d$.

It is clear that the supremum (least upper bound) and the infimum (greatest lower bound) are unique if they exist for a subset of some real numbers.

Theorem 3. *Let S be a nonempty subset of $\mathbb{R}$. If S has an upper bound, then it has a least upper bound (supremum). Similarly, if S has a lower bound, then it has a greatest lower bound (infimum).*

Proof. Since the uniqueness is clear, we only need to show the existence. Note that S is bounded above. For each member $s \in S$, consider its decimal expression

$$s = s_0.s_1s_2s_3\cdots$$

Let S_0 be the set of integer parts of all numbers in S, that is,

$$S_0 := \{s_0 \mid s = s_0.s_1s_2s_3\cdots \in S\}.$$

Clearly, S is nonempty and bounded above (any upper bound for S is an upper bound for S_0). Then S_0 has its largest integer d_0. Let S_1 be the set of the 1st decimal parts of those numbers in S having integer part d_0, that is,

$$S_1 := \{s_1 \mid s = d_0.s_1s_2s_3\cdots \in S\}.$$

Clearly, S_1 is nonempty (because d_0 is the integer part of a number in S) and bounded above. Then S_1 has the largest digit d_1. We define

$$S_2 = \{s_2 \mid s = d_0.d_1s_2s_3\cdots \in S\}.$$

Then S_2 has the largest digit d_2. Continue this procedure; we obtain a real number

$$d = d_0.d_1d_2d_3\cdots$$

We claim that d is the supremum (least upper bound) for S.

First, we show that d is an upper bound. Let $s \in S$ be any member with decimal express

$$s = s_0.s_1s_2s_3\cdots$$

If $s \neq d$, let k be the first decimal place where s and d disagree. Then

$$s = d_0.d_1d_2\cdots d_{k-1}s_ks_{k+1}\cdots, \quad s_k \neq d_k \ \text{ (possibly } k=0).$$

By our choice of d_k, we must have $s_k < d_k$, since

$$S_k := \{s_k \mid s = d_0.d_1d_2\cdots d_{k-1}s_ks_{k+1}\cdots \in S\}, \quad d_k = \max(S).$$

This means that $s < d$. Hence d is an upper bound for S.

Now let u be an upper bound for S with decimal expression

$$u = u_0.u_1u_2u_3\cdots.$$

We need to show that $d \leq u$. If $d \neq u$, let j be the first decimal place where d and u disagree. Then

$$u = d_0.d_1d_2\cdots d_{j-1}u_ju_{j+1}\cdots, \quad u_j \neq d_j.$$

By the choice of d_j, there is a member $s \in S$ with decimal expression

$$s = d_0.d_1d_2\cdots d_{j-1}d_js_{j+1}\cdots$$

Since u is an upper bound for S, we have $s \leq u$. Thus $d_j \leq u_j$. Since $d_j \neq u_j$, we must have $d_j < u_j$. Hence $d < u$. By definition of least upper bound, the real number d is the least upper bound for S. We finish the proof of the first part.

For the second part of the theorem, let S be a nonempty subset of real numbers and is bounded below, that is, there is a real number l such that $s \geq l$ for all $s \in S$. Then $-s \leq -l$ for all $s \in S$. So the set

$$-S := \{-s \mid s \in S\}$$

is nonempty and bounded above. Thus $-S$ has a supremum (least upper bound) d. We shall see that $-d$ is the infimum (greatest lower bound) for S. In fact, since $-s \leq d$ for all $s \in S$, then $s \geq -d$; so $-d$ is a lower bound for S. For any lower bound m of S, i.e., $s \geq m$ for all $s \in S$; so $-s \leq -m$ for all $s \in S$. Thus $-m$ is an upper bound for $-S$. Since d is a supremum (least upper bound) for $-S$, then $d \leq -m$. Hence $-d \geq m$. This means that $-d$ is an infinmum (greatest lower bound) for S. $\square$

Example 2. $e = \lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$.

We shall see below that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is increasing and bounded above. So the set $S = \left\{ \left(1 + \frac{1}{n}\right)^n \,\middle|\, n = 1, 2, \ldots \right\}$ has supremum, denoted e. In fact, applying Binomial Theorem, we have

$$\begin{aligned}
a_n &= 1 + \frac{n}{1!} \cdot \frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!}\left(\frac{1}{n}\right)^n \\
&= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\
&< 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n-1}{n+1}\right) \\
&< 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n-1}{n+1}\right) \\
&\quad + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right) = a_{n+1}.
\end{aligned}$$

Moreover, we see that

$$\begin{aligned}
a_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\
&< 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \quad (\text{notice } 2^{n-1} \leq n! \text{ for } n \geq 2) \\
&\leq 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1 - (1/2)^n}{1 - 1/2} < 3.
\end{aligned}$$

2 Existence of nth Roots

Example 3. There exists a real number c such that $c^3 = 2$.

Proof. The key idea is to define the following set of real numbers

$$S = \{x \mid x \in \mathbb{R}, x^3 < 2\}.$$

It is clear that S is nonempty as 0 and 1 are contained in S. The set S also has an upper bound. For instance, 4 is an upper bound for S. (Otherwise, if $x \in S$ and $x > 4$, then $x^3 > 4^3 = 64$, a contradiction.) Therefore, by Theorem 3, S has a least upper bound; say, $c = \mathrm{LUB}(S)$. Then $c \geq 1$. We claim that $c^3 = 2$. We show the claim by contradiction.

Suppose $c^3 \neq 2$. Then we either have $c^3 < 2$ or $c^3 > 2$. We shall obtain a contradiction for each of the two cases.

CASE 1. Assume $c^3 < 2$. Our strategy is to find a real number $\alpha > 0$ such that $(c+\alpha)^3 < 2$. If so we have $c + \alpha \in S$. This contradicts to that c is an upper bound for S. To find such an α, consider the following argument

$$\begin{aligned}
(c+\alpha)^3 < 2 \quad &\Leftarrow \quad c^3 + 3c^2\alpha + 3c\alpha^2 + \alpha^3 < 2 \\
&\Leftarrow \quad 3c^2\alpha + 3c\alpha^2 + \alpha^3 < 2 - c^3 \\
&\Leftarrow \quad 3c^2\alpha + 3c\alpha + \alpha < 2 - c^3 \quad \text{and} \quad 0 < \alpha < 1 \\
&\Leftarrow \quad \alpha(3c^2 + 3c + 1) < 2 - c^3 \quad \text{and} \quad 0 < \alpha < 1.
\end{aligned}$$

[Since $0 < \alpha < 1$, then $\alpha^2 < \alpha$ and $\alpha^3 < \alpha$.] Since $2 - c^3 > 0$, we may choose α such that

$$0 < \alpha < 1 \quad \text{and} \quad \alpha < \frac{2 - c^3}{3c^2 + 3c + 1}.$$

With such chosen α we have $(c+\alpha)^3 < 2$, which leads to a contradiction as explained above.

CASE 2. Assume $c^3 > 2$. The strategy is to find a real number $\beta > 0$ such that $(c-\beta)^3 > 2$. If so then $x^3 < 2 < (c-\beta)^3$ for all $x \in S$. Subsequently, $x < c - \beta$ for all $x \in S$. This means that $c - \beta$ is an upper bound for S, contradicting to that c is the least upper bound. To find such β, consider the following argument

$$\begin{aligned}
(c-\beta)^3 > 2 \quad &\Leftarrow \quad c^3 - 3c^2\beta + 3c\beta^2 - \beta^3 > 2 \\
&\Leftarrow \quad 3c^2\beta - 3c\beta^2 + \beta^3 < c^3 - 2 \\
&\Leftarrow \quad 3c^2\beta + 3c\beta + \beta < c^3 - 2 \quad \text{and} \quad 0 < \beta < 1 \\
&\Leftarrow \quad \beta(3c^2 + 3c + 1) < 2 - c^3 \quad \text{and} \quad 0 < \beta < 1.
\end{aligned}$$

Since $c^3 - 2 > 0$, we may choose β such that

$$0 < \beta < 1 \quad \text{and} \quad \beta < \frac{c^3 - 2}{3c^2 + 3c + 1}.$$

With such chosen α we have $(c-\beta)^3 < 2$, which leads to a contradiction as explained above. $\square$

Lemma 4. *If $0 < q < \frac{1}{2}$, then for all integers $n \geq 1$*

(a) $(1-q)^n \geq 1 - nq$*;*

(b) $(1+q)^n \leq 1 + 2^n q$*.*

Proof. (a) We have seen by induction on n that $(1+p)^n \geq 1 + np$ when $p \geq -1$. The inequality follows immediately since $-q \geq -\frac{1}{2} > -1$.

(b) We prove the inequality by induction on n. For $n = 1$ it is true as we trivially have the inequality $1 + q \leq 1 + 2q$. Suppose it is true for the case n. Consider the case $n+1$; we have

$$\begin{aligned}
(1+q)^{n+1} \quad &\leq \quad (1 + 2^n q)(1+q) = 1 + 2^n q + q + 2^n q^2 \\
&\leq \quad 1 + 2^n q + 2^n q = 1 + 2^{n+1} q.
\end{aligned}$$

$\square$

Lemma 5. *Let $y > 0$ and $0 < \alpha < \frac{y}{2}$. Then for all integers $n \geq 1$,*

(a) $(y-\alpha)^n \geq y^n - ny^{n-1}\alpha$;

(b) $(y+\alpha)^n \leq y^n + 2^n y^{n-1}\alpha$.

Proof. (a) Since $\frac{\alpha}{y} < \frac{1}{2}$, it follows from Lemma 4(a) that

$$(y-\alpha)^n = y^n\left(1-\frac{\alpha}{y}\right)^n \geq y^n\left(1-\frac{n\alpha}{y}\right) = y^n - ny^{n-1}\alpha.$$

(b) Since $\frac{\alpha}{y} < \frac{1}{2}$, it follows from Lemma 4(b) that

$$(y+\alpha)^n = y^n\left(1+\frac{\alpha}{y}\right)^n \leq y^n\left(1+2^n\frac{\alpha}{y}\right) = y^n + 2^n y^{n-1}\alpha.$$

□

Proposition 6. *Let n be an integer such that $n \geq 2$. If b is a positive real number, then there exists exactly one positive real number c such that*

$$c^n = b.$$

That is, the equation $x^n = b$ has a unique positive real solution.

Proof. Consider the set $S = \{s \in \mathbb{R} \mid s^n < b\}$. It is clear that S is nonempty because $0 \in S$ and S contains a positive real number. In fact, if $b < 1$, then $b^n < b$; thus $b \in S$. If $b \geq 1$, then $\frac{b}{b+1} < 1$; thus $(\frac{b}{b+1})^n < \frac{b}{b+1} < b$; so $\frac{b}{b+1} \in S$.

The set S is bounded above. In fact, if $b \geq 1$, then $b^n \geq x$; thus $s^n < b \leq b^n$ for all $s \in S$; so $s < b$ for all $b \in S$, i.e., b is an upper bound for S. If $x < 1$, then $s^n < b < 1^n$ for all $s \in S$; thus $s < 1$ for all $s \in S$, i.e., 1 is an upper bound for S.

Now by Theorem 3, the set S has a least upper bound. Let $c = \mathrm{LUB}(S)$. Since S contains positive real numbers, it follows that $c > 0$, for y is an upper bound of S. We claim that $c^n = b$. Suppose $c^n \neq b$, then either $c^n < b$ or $c^n > b$.

Case 1: $c^n < b$.

We shall see that there exists a real number $\alpha > 0$ such that $(c+\alpha)^n < b$. If so, we have $c + \alpha \in S$. Since c is an upper bound for S, then $c + \alpha \leq c$, a contradiction. To find such α, notice the following argument by Lemma 5(b):

$$\begin{aligned}(c+\alpha)^n < b \quad &\Leftarrow \quad c^n + 2^n c^{n-1}\alpha < b \quad \text{and} \quad 0 < \alpha < c/2 \\ &\Leftarrow \quad 2^n c^{n-1}\alpha < b - c^n \quad \text{and} \quad 0 < \alpha < c/2.\end{aligned}$$

Since $b - c^n > 0$, we may choose α such that

$$0 < \alpha < \frac{c}{2} \quad \text{and} \quad \alpha < \frac{b-c^n}{2^n c^{n-1}}.$$

Case 2: $c^n > b$.

We shall find a real number $\alpha > 0$ such that $(c-\alpha)^n > b$. Then $s^n < b < (c-\alpha)^n$ for all $s \in S$. It follows that $s < c - \alpha$ for all $s \in S$. This means that $c - \alpha$ is an upper bound of S, a contradiction. To find such α, notice the following argument by Lemma 5(a):

$$\begin{aligned}(c-\alpha)^n > b \quad &\Leftarrow \quad c^n - nc^{n-1}\alpha > b \quad \text{and} \quad 0 < \alpha < c/2 \\ &\Leftarrow \quad nc^{n-1}\alpha < c^n - b \quad \text{and} \quad 0 < \alpha < c/2.\end{aligned}$$

Since $y^n - x > 0$, we may choose α such that

$$0 < \alpha < \frac{c}{2} \quad \text{and} \quad \alpha < \frac{c^n - x}{nc^{n-1}}.$$

□

Exercise 1. Use GLB to show that for any two real numbers $a, b \in \mathbb{R}$, the addition $a + b$, multiplication ab, and $\frac{1}{b}$ (if $b \neq 0$) exist. (Hint: Write a, b in decimal expressions, say, $a = a_0.a_1a_2\cdots$ and $b = b_0.b_1b_2\cdots$. For the addition, one may consider the set

$$S = \left\{ a_0 + b_0 + \frac{a_1 + b_1}{10} + \cdots + \frac{a_k + b_k}{10^k} \;\middle|\; k = 1, 2, \ldots \right\}.$$