Week 5-6: The Binomial Coefficients

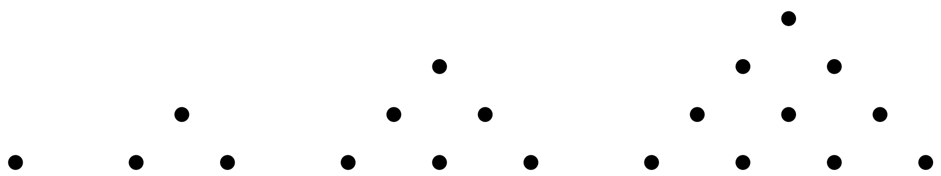
March 6, 2018

1 Pascal Formula

Theorem 1.1 (Pascal's Formula)**.** *For integers n and k such that $n \geq k \geq 1$,*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The numbers $\binom{n}{2} = \frac{n(n-1)}{2}$ $(n \geq 2)$ are **triangle numbers**, that is,



The **pentagon numbers** are $1, 5, 12, 22, \ldots$, defined as the numbers of points of dilated pentagons. Then $a_n = a_{n-1} + 3n + 1$ for $n \geq 1$ with $a_0 = 1$. Then $a_n = \frac{3}{2}n^2 + \frac{5}{2}n + 1$, $n \geq 1$. The k-gon numbers are $1, k, 3k-3, 6k-8, \ldots$.

The numbers $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ $(n \geq 3)$ are **tetrahedral numbers**, i.e., $\binom{n}{3}$ is the number of lattice points of the tetrahedron $\Delta^3(n)$ defined by

$$\Delta^3(n) = \{(x, y, z) : x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq n - 3\}.$$

Theorem 1.2. *The number of nondecreasing coordinate paths from $(0, 0)$ to (m, n) with $m, n \geq 0$ equals*

$$\binom{m+n}{m}.$$

2 Binomial Theorem

Theorem 2.1 (Binomial Expansion)**.** *For integer $n \geq 1$ and variables x and y,*

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

3 Binomial Identities

Definition 3.1. For any real number α and integer k, define the **binomial coefficients**

$$\binom{\alpha}{k} = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k = 0 \\ \alpha(\alpha-1)\cdots(\alpha-k+1)/k! & \text{if } k > 0 \end{cases}$$

Proposition 3.2. *(1) For real number α and integer k,*

$$\binom{\alpha}{k} = \binom{\alpha-1}{k} + \binom{\alpha-1}{k-1}.$$

(2) For real number α and integer k,

$$k\binom{\alpha}{k} = \alpha\binom{\alpha-1}{k-1}.$$

(3) For nonnegative integers m, n, and k such that $m+n \geq k$,

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

Proposition 3.3. *For integers $n, k \geq 0$,*

$$\binom{n+1}{k+1} = \sum_{m=0}^{n} \binom{m}{k}$$

Proof. Applying the Pascal formula again and again, we have

$$\begin{aligned} \binom{n+1}{k+1} &= \binom{n}{k+1} + \binom{n}{k} \\ &= \binom{n-1}{k+1} + \binom{n-1}{k} + \binom{n}{k} \\ &= \cdots \\ &= \binom{0}{k+1} + \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k}. \end{aligned}$$

Note that $\binom{0}{k+1} = 0$. □

4 Multinomial Theorem

Theorem 4.1 (Multinomial Expansion)**.** *For any positive integer n,*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_1,n_2,\ldots,n_k \geq 0}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where the coefficients

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

are called **multinomial coefficients***.*

Proof.

$$\begin{aligned}
(x_1 + x_2 + \cdots + x_k)^n &= \underbrace{(x_1 + x_2 + \cdots + x_k) \cdots (x_1 + x_2 + \cdots + x_k)}_{n} \\
&= \sum u_1 u_2 \cdots u_n \quad (u_i = x_1, x_2 \ldots, x_k,\ 1 \le i \le n) \\
&= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_1, n_2, \ldots, n_k \ge 0}} \left\{ \begin{array}{c} \text{number of permutations of the} \\ \text{multiset } \{n_1 x_1, n_2 x_2, \ldots, n_k x_k\} \end{array} \right\} \\
&= \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_1, n_2, \ldots, n_k \ge 0}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.
\end{aligned}$$

□

5 Newton Binomial Theorem

Theorem 5.1 (Newton's Binomial Expansion)**.** *Let α be a real number. If $0 \le |x| < |y|$, then*

$$(x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha - k},$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}.$$

Proof. Apply the Taylor expansion formula for the function $(x+y)^\alpha$ of two variables. □

Corollary 5.2. *If $|z| < 1$, then*

$$\begin{aligned}
(1 + z)^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k, \\
\frac{1}{(1 - z)^\alpha} &= \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-z)^k = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} z^k.
\end{aligned}$$

The identity

$$\binom{-\alpha}{k} = (-1)^k \binom{\alpha + k - 1}{k}.$$

is called the **reciprocity law** *of binomial coefficients.*

Proof. Apply the Taylor expansion formula. $\square$

In particular, since $\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i$, we have

$$\begin{aligned} \frac{1}{(1-z)^n} &= \left(\sum_{i_1=0}^{\infty} z^{i_1}\right) \cdots \left(\sum_{i_n=0}^{\infty} z^{i_n}\right) \\ &= \sum_{k=0}^{\infty} z^k \sum_{i_1+\cdots+i_n=k} 1 \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k. \end{aligned}$$

This shows again that the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = k$$

equals the binomial coefficient

$$\left\langle {n \atop k} \right\rangle = \binom{n+k-1}{k}.$$

6 Unimodality of Binomial Coefficients

Definition 6.1. A sequence $s_0, s_1, s_2, \ldots, s_n$ is said to be **unimodal** if there is an integer k $(0 \le k \le n)$ such that

$$s_0 \le s_1 \le \cdots \le s_k \ge s_{k+1} \ge \cdots \ge s_n.$$

Theorem 6.2. *Let n be a positive integer. The sequence of binomial coefficients*

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

is an unimodal sequence. More precisely, if n is even,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{n/2} > \cdots > \binom{n}{n-1} > \binom{n}{n};$$

and if n is odd,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2} > \cdots > \binom{n}{n-1} > \binom{n}{n}.$$

Proof. Note that the quotient

$$\binom{n}{k} \Big/ \binom{n}{k-1} = \frac{n-k+1}{k} = \begin{cases} \ge 1 & \text{if } k \le (n+1)/2 \\ \le 1 & \text{if } k \ge (n+1)/2 \end{cases}$$

The unimodality follows immediately. $\square$

A sequence $s_0, s_1, \ldots, s_n$ of positive numbers is said to be **log-concave** if

$$s_i^2 \geq s_{i-1}s_{i+1}, \quad i = 1, \ldots, n-1.$$

The condition implies that the sequence $\log s_1, \log s_2, \ldots, \log s_n$ are concave, i.e.,

$$\log s_i \geq (\log s_{i-1} + \log s_{i+1})/2.$$

Proposition 6.3. *If a sequence (s_i) is log-concace, then it is unimodal.*

Proof. Assume the sequence is nonzero. The condition $s_i^2 \geq s_{i-1}s_{i+1}$ is equivalent to

$$\frac{s_{i-1}}{s_i} \leq \frac{s_i}{s_{i+1}}.$$

If there exists an i_0 such that $s_{i_0} \leq s_{i_0+1}$, i.e., $\frac{s_{i_0}}{s_{i_0+1}} \leq 1$, then $\frac{s_{i-1}}{s_i} \leq 1$ for all $i \leq i_0$, i.e.,

$$s_0 \leq s_1 \leq \cdots \leq s_{i_0} \leq s_{i_0+1}.$$

If there exists an i_0 such that $s_{i_0-1} \geq s_{i_0}$, i.e., $\frac{s_{i_0-1}}{s_{i_0}} \geq 1$, then then $\frac{s_{i-1}}{s_i} \geq 1$ for all $i \geq i_0$, i.e.,

$$s_{i_0-1} \geq s_{i_0} \geq \cdots \geq s_{n-1} \geq s_n.$$

Now for the nondecreasing numbers $\frac{s_i}{s_{i+1}}$, there exists an index i_0 such that

$$\frac{s_{i_0-1}}{s_{i_0}} \leq 1 \leq \frac{s_{i_0}}{s_{i_0+1}}.$$

It follows that

$$s_0 \leq s_1 \leq \cdots \leq s_{i_0} \geq s_{i_0+1} \geq \cdots \geq s_n.$$

□

The sequence $s_i = \binom{n}{i}$ of binomial coefficients is log-concave. In fact,

$$\frac{s_i^2}{s_{i-1}s_{i+1}} = \frac{(n-i+1)(i+1)}{i(n-i)} > 1, \quad i = 1, \ldots, n-1.$$

Given a graph G with n vertices. A coloring of G with t colors is said to be *proper* if no two adjacent vertices receive the same color. The number of proper colorings turns out to be a polynomial function of t, called the chromatic polynomial of G, denoted $\chi(G,t)$, and it can be written as the form

$$\chi(G,t) = \sum_{k=0}^{n} (-1)^{n-k} a_k t^k.$$

Conjecture 6.4 (Log-Concavity Conjecture)**.** *The coefficients of the above chromatic polynomial satisfies the log-concave equality:*

$$a_k^2 \geq a_{k-1}a_{k+1}.$$

When the inequalities are strict inequalities, it is called the Strict Log-Concavity Conjecture.

A **cluster** of a set S is a collection $\mathcal{A}$ of subsets of S such that no one is contained in another. A **chain** is a collection $\mathcal{C}$ of subsets of S such that for any two subsets, one subset is always contained in another subset. For example, for $S = \{a, b, c, d\}$, the collection

$$\mathscr{A} = \Big\{\{a, b\},\ \{b, c, d\},\ \{a, c\},\ \{a, d\}\Big\}$$

is a cluster; while the collection

$$\mathscr{C} = \Big\{\varnothing,\ \{b, d\},\ \{a, b, d\},\ \{a, b, c, d\}\Big\}$$

is a chain. In more general context, a cluster is an antichain of a partially ordered set.

Theorem 6.5 (Sperner)**.** *Every cluster of an n-set S contains at most $\binom{n}{\lfloor n/2 \rfloor}$ subsets of S.*

Proof. Let $S = \{1, 2, \ldots, n\}$. We actually prove the following stronger result by induction on n:

The power set $P(S)$ can be partitioned into disjoint chains $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$ with

$$m = \binom{n}{\lfloor n/2 \rfloor}.$$

If so, then for each cluster $\mathscr{A}$ of S,

$$|\mathscr{A} \cap \mathscr{C}_i| \le 1 \quad \text{for all} \quad 1 \le i \le m.$$

Consequently,

$$|\mathscr{A}| = \left|\mathscr{A} \cap \bigcup_{i=1}^{m} \mathscr{C}_i\right| = \sum_{i=1}^{m} |\mathscr{A} \cap \mathscr{C}_i| \le m = \binom{n}{\lfloor n/2 \rfloor}.$$

For $n = 1$, $\binom{n}{\lfloor n/2 \rfloor} = \binom{1}{0} = 1$,

$$\varnothing \subset \{1\}.$$

For $n = 2$, $\binom{n}{\lfloor n/2 \rfloor} = \binom{2}{1} = 2$,

$$\varnothing \subset \{1\} \subset \{1, 2\},$$
$$\{2\}.$$

For $n = 3$, $\binom{n}{\lfloor n/2 \rfloor} = \binom{3}{1} = 3$,

$$\varnothing \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\},$$
$$\{2\} \subset \{2, 3\},$$
$$\{3\} \subset \{1, 3\}.$$

For $n=4$, $\binom{n}{\lfloor n/2\rfloor}=\binom{4}{2}=6$. The 6 chains can be obtained in two ways: (i) Attach a new subset at the end to each chain of the chain partition for $n=3$ (this new subset is obtained by appending 4 to the last subset of the chain); (ii) delete the last subsets in all chains of the partition for $n=3$ and append 4 to all the remaining subsets.

$$\begin{gathered}
\varnothing \subset \{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\}, \\
\{2\} \subset \{2,3\} \subset \{2,3,4\}, \\
\{3\} \subset \{1,3\} \subset \{1,3,4\}, \\
\{4\} \subset \{1,4\} \subset \{1,2,4\}, \\
\{2,4\}, \\
\{3,4\}.
\end{gathered}$$

Note that the chain partition satisfies the properties: (i) Each chain is saturated in the sense that no subset can be added in between any two consecutive subsets; (ii) in each chain the size of the beginning subset plus the size of the ending subset equals n. A chain partition satisfying the two properties is called a **symmetric chain partition**. The above chain partitions for $n=1,2,3,4$ are symmetric chain partitions.

Given a symmetric chain partition for the case $n-1$; we construct a symmetric chain partition for the case n: For each chain $A_1\subset A_2\subset\cdots\subset A_k$ in the chain partition for the case $n-1$,

$$\begin{aligned}
\text{if } k\ge 2,\ \text{ do }\ & A_1\subset A_2\subset\cdots\subset A_k\subset A_k\cup\{n\}, \quad \text{and} \\
& A_1\cup\{n\}\subset A_2\cup\{n\}\subset\cdots\subset A_{k-1}\cup\{n\}; \\
\text{if } k=1,\ \text{ do }\ & A_1\subset A_1\cup\{n\}.
\end{aligned}$$

It is clear that the chains constructed form a symmetric chain partition. In fact, the chains constructed are obviously saturated. Since $|A_1|+|A_k|=n-1$, then $|A_1|+|A_k\cup\{n\}|=|A_1|+|A_k|+1=n$, and when $k\ge 2$,

$$|A_1\cup\{n\}|+|A_{k-1}\cup\{n\}|=|A_1|+|A_{k-1}|+2=|A_1|+|A_k|+1=n.$$

Now for each chain $B_1\subset B_2\subset\cdots\subset B_l$ of the symmetric chain partition for the case n, since $|B_1|\le|B_l|$, we must have $|B_1|\le n/2\le|B_l|$ (otherwise, if $|B_l|<n/2$ then $|B_1|+|B_2|<n$, or if $|B_1|>n/2$ then $|B_1|+|B_l|>n$). By definition of $\lfloor n/2\rfloor$ and $\lceil n/2\rceil$, we have

$$|B_1|\le\lfloor n/2\rfloor\le\lceil n/2\rceil\le|B_l|.$$

This means that $B_1\subset B_2\subset\cdots\subset B_l$ contains exactly one $\lfloor n/2\rfloor$-subset and exactly one $\lceil n/2\rceil$-subset. Note that the number of $\lfloor n/2\rfloor$-subsets of S is $\binom{n}{\lfloor n/2\rfloor}$ and the number of $\lceil n/2\rceil$-subsets of S is $\binom{n}{\lceil n/2\rceil}$. It follows that the number of chains in the constructed symmetric chain partition is

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}.$$

Thus every cluster of the power set $P(S)$ has size less than or equal to $\binom{n}{\lfloor n/2 \rfloor}$. The cluster $P_{\lfloor n/2 \rfloor}(S)$ is of size $\binom{n}{\lfloor n/2 \rfloor}$. $\square$

The proof of the Spencer theorem actually gives the construction of clusters of maximal size. When $n = \mathit{even}$, there is only one such cluster,

$$P_{\frac{n}{2}}(S): \quad \text{the collection of all } \tfrac{n}{2}\text{-subsets of } S;$$

and when $n = \mathit{odd}$, there are exactly two such clusters,

$$P_{\frac{n-1}{2}}(S): \text{ the collection of all } \tfrac{n-1}{2}\text{-subsets of } S, \text{ and}$$
$$P_{\frac{n+1}{2}}(S): \text{ the collection of all } \tfrac{n+1}{2}\text{-subsets of } S.$$

Example 6.1. (a) Let $S = \{1\}$. Then $n = 1$ and $\binom{1}{0} = \binom{1}{1} = 1$. There are two clusters: $\varnothing$ and $\{1\}$.

(b) Let $S = \{1, 2\}$. Then $n = 2$ and $\binom{2}{1} = 2$. There is only one cluster of maximal size: $\big\{\{1\}, \{2\}\big\}$.

(c) Let $S = \{1, 2, 3\}$. Then $n = 3$ and $\binom{3}{1} = \binom{3}{2} = 3$. There are two clusters of maximal size:

$$\big\{\{1\},\ \{2\},\ \{3\}\big\} \quad \text{and} \quad \big\{\{1,2\},\ \{1,3\},\ \{2,3\}\big\}$$

(d) Let $S = \{1, 2, 3, 4\}$. Then $n = 4$ and $\binom{4}{2} = 6$. There is only one cluster of maximal size:

$$\big\{\{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\}\big\}$$

(e) Let $S = \{1, 2, 3, 4, 5\}$. Then $n = 5$ and $\binom{5}{2} = \binom{5}{3} = 10$. There are two clusters of maximal size:

$$\Big\{\{1,2\},\ \{1,3\},\ \{1,4\},\ \{1,5\},\ \{2,3\},\ \{2,4\},\ \{2,5\},\ \{3,4\},\ \{3,5\},\ \{4,5\}\Big\},$$

$$\Big\{\{1,2,3\},\ \{1,2,4\},\ \{1,2,5\},\ \{1,3,4\},\ \{1,3,5\},\ \{1,4,5\},\ \{2,3,4\},\ \{2,3,5\},\ \{2,4,5\},\ \{3,4,5$$

7 Dilworth Theorem

Let $(X, \leq)$ be a finite partially ordered set. A subset A of X is called an **antichain** if any two elements of A are incomparable. In contrast, a **chain** is a subset C of X whose any two elements are comparable. Thus a chain is a linearly ordered subset of X. It is clear that any subset of a chain is also a chain, and any subset of an antichain is also an antichain. The important connection between chains and antichains is:

$$|A \cap C| \leq 1 \quad \text{for any antichain } A \text{ and chain } C.$$

Example 7.1. Let $X = \{1, 2, \ldots, 10\}$. The divisibility $|$ makes X into a partially ordered set. The subsets

$$\{2,3,5,7\},\ \{2,5,7,9\},\ \{3,4,5,7\},\ \{3,4,7,10\},\ \{3,5,7,8\},\ \{3,7,8,10\},$$

$$\{4,5,6,7,9\},\ \{4,6,7,9,10\},\ \{5,6,7,8,9\},\ \{6,7,8,9,10\}$$

are antichains, they are actually maximal antichains; while the subsets

$$\{1,2,4,8\}, \quad \{1,3,6\}, \quad \{1,3,9\}, \quad \{1,5,10\}, \quad \{1,7\}$$

are chains and they are actually maximal chains.

Let $(X, \leq)$ be a finite poset. We are interested in partitioning X into disjoint union of antichains and partitioning X into disjoint union of chains. Let $\mathcal{A}$ be an antichain partition of X and let C be a chain of X. Since no two elements of C can be contained in any antichain in $\mathcal{A}$, then

$$|\mathcal{A}| \geq |C|.$$

Similarly, for any chain partition $\mathcal{C}$ and an antichain A of X, there are no two elements of A belonging to a chain of $\mathcal{C}$, we then have

$$|\mathcal{C}| \geq |A|.$$

Theorem 7.1. *Let $(X, \leq)$ be a finite poset, and let r be the largest size of a chain. Then X can be partitioned into r but no fewer antichains. In other words,*

$$\min\big\{|\mathcal{A}| : \mathcal{A} \text{ is an antichain partition}\big\} = \max\big\{|C| : C \text{ is a chain}\big\}.$$

Proof. It is enough to show that X can be partitioned into r antichains. Let $X_1 = X$ and let A_1 be the set of all minimal elements of X_1. Let $X_2 = X_1 - A_1$ and let A_2 be the set of all minimal elements of X_2. Let $X_3 = X_2 - A_2$ and let A_3 be the set of all minimal elements of X_3. Continuing this procedure we obtain a decomposition of X into antichains $A_1, A_2, \ldots, A_p$. By the previous argument we always have $p \geq r$. On the other hand, for any $a_p \in A_p$, there is an element $a_{p-1} \in A_{p-1}$ such that $a_{p-1} < a_p$. Similarly, there is an element $a_{p-2} \in A_{p-2}$ such that $a_{p-2} < a_{p-1}$. Continuing this process we obtain a chain $a_1 < a_2 < \cdots < a_p$. Since r is the largest size of a chain, we then have $r \geq p$. Thus $p = r$. $\square$

The following dual version of the theorem is known as the **Dilworth Theorem**.

Theorem 7.2 (Dilworth). *Let $(X, \leq)$ be a finite poset. Let s be the largest size of an antichain. Then X can be partitioned into s, but not less than s, chains. In other words,*

$$\min\left\{|\mathcal{C}| : \mathcal{C} \text{ is a chain partition}\right\} = \max\left\{|A| : A \text{ is an antichain}\right\}.$$

Proof. It suffices to show that X can be partitioned into s chains. We proceed by induction on $|X|$. Let $|X| = n$. For $n = 1$, it is trivially true. Assume that $n \geq 2$. Let $A_{\min}$ be the set of all minimal elements of X, and $A_{\max}$ the set of all maximal elements of X. Both $A_{\min}$ and $A_{\max}$ are maximal antichains. We divide the situation into two cases.

Case 1. $A_{\min}$ and A_{man} are the only maximal antichains of X. Take an element $x \in A_{\min}$ and an element $y \in A_{\text{man}}$ such that $x \leq y$ (possibly $x = y$). Let $X' = X - \{x, y\}$. If $X' = \varnothing$, then $X = \{x, y\}$ and $x < y$, thus $s = 1$ and $x < y$ is the required chain partition. Assume $X' \neq \varnothing$, then X' has only the maximal antichains $A_{\min} - \{x\}$ and $A_{\max} - \{y\}$. The largest size of antichains of X' is $s - 1$. Since $|X'| \leq n - 1$, by induction the set X' can be partitioned into $s - 1$ chains $C_1, \ldots, C_{s-1}$. Set $C_s = \{x \leq y\}$. The collection $\{C_1, \ldots, C_s\}$ is a chain partition of X.

Case 2. The set X has a maximal antichain $A = \{a_1, a_2, \ldots, a_s\}$ of size s such that $A \neq A_{\min}$ and $A \neq A_{\text{man}}$. Let

$$\begin{aligned} A^- &= \{x \in X : x \leq a_i \text{ for some } a_i \in A\}, \\ A^+ &= \{x \in X : x \geq a_i \text{ for some } a_i \in A\}. \end{aligned}$$

The sets A^+ and A^- satisfy the following properties:

1. $A^+ \subsetneq X$. (Since $A_{\min} \not\subseteq A$, i.e., there is a minimal element not in A; this minimal element cannot be in A^+, otherwise, it is larger than one element of A by definition.)

2. $A^- \subsetneq X$. (Since $A_{\max} \not\subseteq A$, i.e., there is a maximal element not in A; this maximal element cannot be in A^-, otherwise, it is smaller than one element of A by definition.)

3. $A^- \cap A^+ = A$. (It is always true that $A \subseteq A^+ \cap A^-$. For each $x \in A^+ \cap A^-$, there exist $a_i, a_j \in A$ such that $a_i \leq x \leq a_j$ by definition, then $a_i \leq a_j$, which implies $a_i = a_j$ so that $i = j$, thus $x = a_i = a_j \in A$.)

4. $A^- \cup A^+ = X$. (Suppose there is an element $x \notin A^- \cup A^+$, then x is neither ahead nor behind any member of A, thus $A \cup \{x\}$ is an antichain of larger size than A.)

Since A^- and A^+ are smaller posets having the maximal antichain A of size s, then by induction, by induction A^- can be partitioned into s chains $C_1^-, C_2^-, \ldots, C_s^-$ with the maximal elements $a_1, a_2, \ldots, a_s$ respectively, and A^+ can be partitioned into s chains $C_1^+, C_2^+, \ldots, C_s^+$ with the minimal elements $a_1, a_2, \ldots, a_s$ respectively. Thus we obtain a partition of X into s chains

$$C_1^- \cup C_1^+, \quad C_2^- \cup C_2^+, \quad \ldots, \quad C_s^- \cup C_s^+.$$

□

Example 7.2. Let $X = \{1, 2, \dots, 20\}$ be the poset with the partial order of divisibility. Then the subset $\{1, 2, 4, 8, 16\}$ is a chain of maximal size. The set X can be partitioned into five antichains

$$\{1\}, \quad \{2, 3, 5, 7, 11, 13, 17, 19\}, \quad \{4, 6, 9, 10, 14, 15\}, \quad \{8, 12, 18, 20\}, \quad \{16\}.$$

However, the size of the antichain $\{2, 3, 5, 7, 11, 13, 17, 19\}$ of size 8 is not maximal. In fact,

$$\{4, 6, 7, 9, 10, 11, 13, 15, 17, 19\}$$

is an antichain of size 10. The set X can be partitioned into ten chains

$$\{1, 2, 4, 8, 16\}, \quad \{3, 6, 12\}, \quad \{5, 10, 20\}, \quad \{7, 14\}, \quad \{9, 18\},$$

$$\{11\}, \quad \{13\}, \quad \{15\} \quad \{17\}, \quad \{19\}.$$

This means that $\{4, 6, 7, 9, 10, 11, 13, 15, 17, 19\}$ is an antichain of maximal size.

Example 7.3. Let $X = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i, j \leq 3, \}$ be a poset whose partial order $\leq$ is defined by $(i, j) \leq (k, l)$ if and only if $i \leq k$ and $j \leq l$. The size of the longest chain is 7. For instance,

$$(0, 0) < (1, 0) < (1, 1) < (1, 2) < (2, 2) < (2, 3) < (3, 3)$$

is a chain of length 7. The the following collection of subsets

$$\{(0, 0)\}, \quad \{(1, 0), (0, 1)\}, \quad \{(2, 0), (1, 1), (0, 2)\}, \quad \{(3, 0), (2, 1), (1, 2), (0, 3)\},$$

$$\{(3, 1), (2, 2), (1, 3)\}, \quad \{(3, 2), (2, 3)\}, \quad \{(3, 3)\}$$

is an antichain partition of X. The maximal size of antichain is 4 and the poset X can be partitioned into 4 disjoint chains:

$$(0, 0) < (0, 1) < (0, 2) < (0, 3) < (1, 3) < (2, 3) < (3, 3),$$

$$(1, 0) < (1, 1) < (1, 2) < (2, 2) < (3, 2),$$

$$(2, 0) < (2, 1) < (3, 1),$$

$$(3, 0).$$

$$\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 3), (2, 3), (3, 3)\},$$

$$\{(1, 0), (1, 1), (1, 2), (2, 2), (3, 2)\},$$

$$\{(2, 0), (2, 1), (3, 1)\},$$

$$\{(3, 0)\}.$$

The Hasse diagram of the poset X is

$$\begin{tikzcd}
 & & & (3,3) \arrow[dl] \arrow[dr] & & & \\
 & & (3,2) \arrow[dl] \arrow[dr] & & (2,3) \arrow[dl] \arrow[dr] & & \\
 & (3,1) \arrow[dl] \arrow[dr] & & (2,2) \arrow[dl] \arrow[dr] & & (1,3) \arrow[dl] \arrow[dr] & \\
(3,0) \arrow[dr] & & (2,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & & (0,3) \arrow[dl] \\
 & (2,0) \arrow[dr] & & (1,1) \arrow[dl] \arrow[dr] & & (0,2) \arrow[dl] & \\
 & & (1,0) \arrow[dr] & & (0,1) \arrow[dl] & & \\
 & & & (0,0) & & &
\end{tikzcd}$$

Finding an antichain of maximal size for a poset is a difficult problem. So far there is no canonical way to do this job.