Homework 1

Chapter 1 (pp.20–25, 4th edition: 2, 4, 8, 13, 30, 33, 35)

1. Consider an m-by-n chessboard with m and n both odd. To fix the notation, suppose that the square in the upper left-hand corner is colored white. Show that if a white square is cut out anywhere on the board, the resulting pruned board has a perfect cover by dominoes.

2. (a) Let $f(n)$ count the number of different perfect covers of a 2-by-n chessboard by dominoes. Evaluate $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$. Try to find (and verify) a simple relation that the counting function f satisfies. use this relation to compute $f(12)$.

   *(b) Let $g(n)$ be the number of different perfect covers of a 3-by-n chessboard by dominoes. Evaluate $g(1), g(2), \ldots, g(6)$.

3. Let a and b be positive integers with a a factor of b. Show that an m-by-n board has a perfect cover by a-by-b pieces if and only if (i) a is a factor of both m and n, and (ii) b is a factor of either m or n. (Hint: Partition the a-by-b pieces into a 1-by-b pieces.)

4. Use de la Loubère's method to construct a magic square of order 7.

5. Is 4-pile Nim game with heaps of sizes 22, 19, 14, and 11 balanced or unbalanced? Player I's first move is to remove 6 coins from the heap of size 19. What should Play II's first move be?

6. Show that in an unbalanced game of Nim in which the largest unbalanced bit is the jth bit, Player I can always balance the game by removing coins from any heap that the base 2 numeral of whose number has a 1 in the jth bit.

7. A game is played between two players, alternating turns as follows: The game starts with an empty pile. When it is his turn a play may add either $1, 2, 3$, or 4 to the pile. The prson who adds the 100th coin to the pile is the winner. Determine whether it is the first or second player who can guarantee a win in this game. What is the winning strategy to follow?

Chapter 2 (pp.40–43, 4th edition: 5, 9, 14, 15, 17)

1. Show that $n + 1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there are always two which differ by at most 2.

2. In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?

3. A bag contains 100 apples, 100 bananas, 100 oranges, and 100 pears. If I pick one piece of fruit out f the bag every minute, how long will it be before I am assured of having picked at least a dozen pieces of fruit of the same kind?

4. Prove that, for any $n + 1$ integers $a_1, a_2, \ldots, a_{n+1}$, there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

5. There are 100 people at a party. Each person has an even number (possibly zero) of acquaintances. Prove that there are three people at the party with the same number of acquaintances.

1. Let r, s, and b be positive integers such that $r \leq s < b$. Then in any b-cyclic coloring of an r-by-s board, there are at least two colors, say c_1 and c_2, such that the number of squares of color c_1 is *not* equal to the number of squares of color c_2.

2. For the game of Nim, let us restrict that each player can move one or two coins. Find the winning strategy for each player.

3. Let n be a positive integer. In the game of Nim let us restrict that each player can move only $i \in \{1, 2, \ldots, n\}$ coins each time from one heap. Find the winning strategy for each player. (Hint: Use modulo integers and write them as base 2 numerals.)

4. Given $m(m-1)^2 + 1$ integral points on a plane, where m is odd. Show that there exists m points whose center is also an integral point.

1 Partial Solutions

No. 4 (p.20) $f(n) = f(n-1) + f(n-2)$.
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No. 8 (p.22)

Since $a|b$, we write $b = ak$. It is clear that if the $m \times n$-board can be covered by $a \times b$-ominoes then it must be covered by $a \times a$-squares. Thus it is necessary that $a|m$ and $a|n$. We divide the $m \times n$-board into $\frac{m}{a} \times \frac{n}{a}$-board so that each big square consists of a^2 small squares. Now the original covering problem becomes the problem to tile the $\frac{m}{a} \times \frac{n}{a}$-board by k-ominoes. It is known that the necessary and sufficient condition is that $k|\frac{m}{a}$ or $k|\frac{n}{a}$. Thus an $m \times n$-board can be tiled by $a \times b$-ominoes if and only if

$$a|m, \ a|n, \text{ and either } b|m \text{ or } b|n.$$

2 The Game of Nim with Restriction

The case of one pile:

$-$	$x : x \equiv 0 (\text{mod } 3)$
$+$	$x : x \not\equiv 0 (\text{mod } 3)$

The case of two piles:

$-$	$(x, y) : x \equiv y (\text{mod } 3)$
$+$	$(x, y) : x \not\equiv y (\text{mod } 3)$

The case of three piles:

$-$	$(x, y, z) : x \equiv 0 (\text{mod } 3), \ y \equiv z (\text{mod } 3)$
$+$	$(x, y, z) : x \not\equiv 0 (\text{mod } 3) \quad \text{or} \quad x \equiv 0 (\text{mod } 3), \ y \not\equiv z (\text{mod } 3)$

The case of n piles: $(a_1, a_2, \ldots, a_n)$

Dividing each pile a_i by 3 to have $a_i = 3q_i + r_i$, $0 \leq r_i < 3$, $1 \leq i \leq n$.

The second player wins: The number of piles having remainder 1 is even, the number of piles having the remainder 2 is even.

The first player wins: Otherwise.

Example 2.1. The game of Nim can be further generalized to that each player can move any amount of coins from the set $\{1, 2, \ldots, n\}$, where $n \geq 1$.

The case of one heap:

$-$	$(x) : x \equiv 0(\mathrm{mod}\ n+1)$
$+$	$(x) : x \not\equiv 0(\mathrm{mod}\ n+1)$

The case of two heaps:

$-$	$(x,y) : x \equiv y(\mathrm{mod}\ n+1)$
$+$	$(x,y) : x \not\equiv y(\mathrm{mod}\ n+1)$

The case of k heaps: $(x_1, \ldots, x_k)$

Let $x_i = q_i(n+1) + r_i$, where $0 \le r_i \le n$ and $1 \le i \le k$. Write $r_1, r_2, \ldots, r_k$ as base 2 numerals:

$$
\begin{aligned}
r_1 &= a_s a_{s-1} \cdots a_2 a_1 a_0, \\
r_2 &= b_s b_{s-1} \cdots b_2 b_1 b_0, \\
&\vdots \\
r_k &= c_s c_{s-1} \cdots c_2 c_1 c_0.
\end{aligned}
$$

Dividing each heap into s subheaps, the the game becomes a game having total ks subheaps:

$$(a_s, \ldots, a_1, a_0; b_s, \ldots, b_1, b_0; \ldots; c_s, \ldots, c_1, c_0).$$

The game of Nim with restriction is called *balanced* if the integers

$$a_s + b_s + \cdots + c_s, \quad \ldots, \quad a_1 + b_1 + \cdots + c_1, \quad a_0 + b_0 + \cdots + c_0$$

are all even; otherwise, it is called *unbalanced*.

When the game is unbalanced, it is always possible to move certain amount of coins in the largest heap so that the game becomes balanced. When the game is unbalanced, player I wins the game since player I can always balance the game. When the game is balanced, player II wins the game. For example, $n = 6$, $(x_1, x_2, x_3, x_4) = (10, 12, 14, 16)$, we have

Size of heaps	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
$10 \equiv 3(\mathrm{mod}\ 7)$	0	1	1
$12 \equiv 5(\mathrm{mod}\ 7)$	1	0	1
$14 \equiv 0(\mathrm{mod}\ 7)$	0	0	0
$16 \equiv 2(\mathrm{mod}\ 7)$	0	1	0

The game is unbalanced. We can take 4 coins away from the heap 2 to balance the game as

Size of heaps	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$
$10 \equiv 3(\mathrm{mod}\ 7)$	0	1	1
$8 \equiv 1(\mathrm{mod}\ 7)$	0	0	1
$14 \equiv 0(\mathrm{mod}\ 7)$	0	0	0
$16 \equiv 2(\mathrm{mod}\ 7)$	0	1	0

$\longrightarrow$