Homework 2

Chapter 3, pp.76: 9, 14, 20, 26, 30, 43.

1. In how many ways can 15 people be seated at a round table if a person B refuses to sit next to a person A? What is B only refuses to sit on A's right?
2. Classroom has 2 rows of 8 seats each. There are 14 students, 5 of them always sit in the front row and 4 of them always sit in the back row. In how many ways can the students be seated?
3. Determine the number of circular permutations of $\{0, 1, 2, \ldots, 9\}$ in which 0 and 9 are not opposite. (Hint: Count those in which 0 and 9 are opposite.)
4. A group of mn people are to be arranged into m teams each with n players.
   (a) Determine the number of ways if each team has a different name.
   (b) Determine the number of ways if the teams don't have names.
5. We are to seat 5 men, 5 women, and 1 dog in a circular arrangement around a round table. In how many ways can this be done if no man is to sit next to a man and no woman is to sit next to a woman?
6. Determine the number of r-combinations of the multiset
$$\{1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}.$$

Chapter 4, p.117: 4, 6, 7, 8, 12, 23, 28, 38, 39.

1. Prove that in the algorithm of generating all permutations of $\{1, 2, \ldots, n\}$, the directions of 1 and 2 never change.
2. Determine the inversion sequences of the following permutations of $\{1, 2, \ldots, 8\}$: 35168274; 83476215.
3. Construct the permutations of $\{1, 2, \ldots, 8\}$ whose inversion sequences are the following: (2,5,5,0,2,1,1,0); (6,6,1,4,2,1,0,0).
4. How many permutations of $\{1, 2, \ldots, 6\}$ have (a) exactly 15 inversions, (b) exactly 14 inversions, and (c) exactly 13 inversions?
5. Let $S = \{x_7, x_6, \ldots, x_1, x_0\}$. Determine the combinations o f S corresponding to the following 8-tuples: (a) 00011011; (b) 01010101; (c) 00001111.
6. Determine the immediate successors of the following 9-tuples in the reflected Gray code of order 9: (a) 010100110; (b) 110001100; (c) 111111111.
7. Determine the 7-combination of $\{1, 2, \ldots, 15\}$ that immediately follows 1,2,4,6,8,14,15 in the lexicographic order. Determine the 7-combination that immediately precedes 1,2,4,6,8,14,15.
8. Let $(X_i, \leq_i)$ be partially ordered sets, $i = 1, 2$. Define a relation T on the set
$$X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$$
by $(x_1, x_2)T(x_1', x_2')$ if and only if $x_1 \leq_1 x_1'$ and $x_2 \leq_2 x_2'$. Prove that $(X_1 \times X_2, T)$ is a partially ordered set. $(X_1 \times X_2, T)$ is called the **direct product** of $(X_1, \leq_1)$ and $(X_2, \leq_2)$ and is also denoted by $(X_1, \leq_1) \times (X_2, \leq_2)$. More generally, prove that the direct product $(X_1, \leq_1) \times (X_2, \leq_2) \times \cdots \times (X_m, \leq_m)$ of partially ordered sets is also a partially ordered set.

9. Let $(J, \leq)$ be the partially ordered set with $J = \{0, 1\}$ and with $0 < 1$. By identifying the combinations of a set X of n elements with the n-tuples of 0's and 1's, prove that the partially ordered set $X, \subseteq)$ can be identified with the n-fold direct product $(J, \leq) \times (J, \leq) \times \cdots \times (J, \leq)$ (n factors).

Supplementary Exercises

1. Find the number of ways to select m numbers from $\{1, 2, \ldots, n\}$ so that no two numbers are consecutive.

Method 1. Let $a_1, a_2, \ldots, a_m$ be such a selection and $a_1 < a_2 < \cdots < a_m$. Let k_i be the number of integers between a_{i-1} and a_i, where $1 \leq i \leq m+1$, $a_0 = 0$ and $a_{m+1} = n+1$. Then the answer is equal to the number of integral solutions of

$$k_1 + k_2 + \cdots + k_{m+1} = n - m$$

satisfying $k_1 \geq 0$, $k_i \geq 1$ for $2 \leq i \leq m$, and $k_{m+1} \geq 0$. This is equivalent to finding the number of nonnegative integral solutions of the equation

$$x_1 + x_2 + \cdots + x_{m+1} = n - m - (m-1) = n - 2m + 1.$$

Then the answer is

$$\left\langle \begin{matrix} m+1 \\ n-2m+1 \end{matrix} \right\rangle = \binom{n-m+1}{n-2m+1} = \binom{n-m+1}{m}.$$

Method 2. Let $a_1, a_2, \ldots, a_m$ be such a selection and $a_1 < a_2 < \cdots < a_m$. If $a_m \neq n$, we change each pair $\{a_i, a_i+1\}$ to a domino and any other number in $\{1, 2, \ldots, n\}$ to a square. Then the selection $\{a_1, a_2, \ldots, a_m\}$ can be viewed as a permutation of dominoes and squares with exactly m dominoes and $n-2m$ squares. There are

$$\binom{n-m}{m}$$

such permutations.

If $a_m = n$, then $a_{m-1} \neq n-1$. This is equivalent to selecting $m-1$ integers from $\{1, 2, \ldots, n-1\}$ such that no two numbers are consecutive and the last number $n-1$ is not selected. There are

$$\binom{n-1-(m-1)}{m-1} = \binom{n-m}{m-1}$$

such selections. Therefore, the answer is to the question

$$\binom{n-m}{m} + \binom{n-m}{m-1} = \binom{n-m+1}{m}.$$

2. A *move* of a permutation of $\{1, 2, \ldots, n\}$ is to take an integer in the permutation and insert it somewhere in the permutation. For example,

$$352614 \to 352146 \to 321456 \to 213456 \to 123456;$$
$$352614 \to 352146 \to 321456 \to 342156 \to 134256 \to 123456.$$

Give an algorithm to compute the minimal number of moves to restore an arbitrary permutation of $\{1, 2, \ldots, n\}$ back to the form $12 \cdots n$.

Solution. For any permutation $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$, we denote by $\delta(a_1 a_2 \cdots a_n)$ the minimal number of moves required to change it back to $12 \cdots n$. Let $n = a_k$ for some k. Notice the following observation: If n must be moved in order to change the permutation back to $12 \cdots n$, then

$$\delta(a_1 a_2 \cdots a_k \cdots a_n) = \delta(a_1 a_2 \cdots a_{k-1} a_{k+1} \cdots a_n) + 1;$$

if n is not necessarily moved to change the permutation back to $12 \cdots n$, then the numbers $a_{k+1}, \ldots, a_n$ must be moved in order to make n in the last position.

In the latter case, we observe that the moves of the numbers $a_{k+1}, \ldots, a_n$ do not change the relative positions of the numbers $a_1, a_2, \ldots, a_{k-1}$ in the permutation $a_1 a_2 \cdots a_{k-1}$. Since the minimal number of moves required to sort $a_1 a_2 \cdots a_{k-1}$ is $\delta(a_1 a_2 \cdots a_{k-1})$, we have

$$\delta(a_1 a_2 \cdots a_k \cdots a_n) = \delta(a_1 a_2 \cdots a_{k-1}) + n - k.$$

Therefore, if $n = a_k$, then we obtain the following recurrence relation

$$\delta(a_1 a_2 \cdots a_k \cdots a_n) = \min\{\delta(a_1 a_2 \cdots a_{k-1} a_{k+1} \cdots a_n) + 1,\ \delta(a_1 a_2 \cdots a_{k-1}) + n - k\}.$$

For example, for the permutation 352614,

$$\begin{aligned}\delta(352614) &= \min\{\delta(35214) + 1,\ \delta(352) + 2\} \\ &= \min\{\delta(3214) + 2,\ \delta(3) + 4,\ \delta(32) + 3, \delta(3) + 3\} = 3.\end{aligned}$$

In fact, the following specific moves sort the permutation:

$$352614 \to 135264 \to 134526 \to 123456.$$

Note that the number 6 should not be moved; otherwise 4 moves are required to sort the permutation. However, the number 6 must be moved in the permutation 136452 to achieve minimality:

$$\begin{aligned}\delta(126453) &= \min\{\delta(12453) + 1,\ \delta(12) + 3\} \\ &= \min\{\delta(1243) + 2,\ \delta(124) + 1 + 1, 3\} = 2.\end{aligned}$$

The actual two moves can be taken as $126453 \to 124536 \to 123456$.