Homework 3

Chapter 5, pp.153: 11, 12, 22, 31, 40, 45.
Chapter 6, p.185: 4, 11, 16, 24, 30.

1. Use combinatorial reasoning to prove the identity

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

Proof. Let S be a set of n elements, and let a, b, c be distinct elements of S. The number of k-subsets of S is $\binom{n}{k}$, and the number of k-subsets of $S - \{a, b, c\}$ is $\binom{n-3}{k}$. Then the LHS is the number of k-subsets of S that contains at least of the elements of $\{a, b, c\}$. Such k-subsets can be divided into 3 types: (1) the k-subsets that contain the element a; (2) the k-subsets that do not contain a but contain b; and (3) the k-subsets that do not contain a, b but contain c. The numbers k-subsets of type (1), type (2), type (3) are

$$\binom{n-1}{k-1}, \quad \binom{n-2}{k-1}, \quad \binom{n-3}{k-1}$$

respectively. The sum of these numbers is exactly the RHS. □

2. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n\text{=odd} \\ (-1)^m \binom{2m}{m} & \text{if } n = 2m \end{cases}$$

Proof. Consider the expansion of $(1+x)^n(1-x)^n = (1-x^2)^n$. On the one hand,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i,$$

$$(1-x)^n = \sum_{i=0}^{n} (-1)^j \binom{n}{j} x^j.$$

Then the coefficient of x^n in the product $\left[\sum_{i=0}^{n} \binom{n}{i} x^i\right] \left[\sum_{i=0}^{n} (-1)^j \binom{n}{i} x^j\right]$ is given by

$$\sum_{i+j=n} (-1)^j \binom{n}{i} \binom{n}{j} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$$

which is exactly the LHS. On the other hand,

$$(1-x^2)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{2i}.$$

There are only even terms in the expansion. Thus the coefficient of x^n is zero if n is odd; and the coefficient of x^n is $(-1)^m \binom{2m}{m}$ if $n = 2m$ is even. □

3. Prove that for all real numbers α and all integers k and n,

$$\binom{\alpha}{n}\binom{n}{k} = \binom{\alpha}{k}\binom{\alpha-k}{n-k}.$$

Proof. For $n < k$, the LHS is zero because $\binom{n}{k} = 0$. The RHS is also zero because $\binom{\alpha-k}{n-k} = 0$ by definition.
For $n \geq k$, we divide the situation into the following cases:
If $k \geq 1$, then

$$\begin{aligned} LHS &= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \cdot \frac{n!}{k!(n-k)!} \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \cdot \frac{(\alpha-k)(\alpha-k-1)\cdots(\alpha-n+1)}{(n-k)!} \\ &= RHS. \end{aligned}$$

If $k = 0$, then both LHS and RHS are both equal to $\binom{\alpha}{n}$ because $\binom{n}{k} = \binom{\alpha}{k} = 1$ by definition.
If $k \leq -1$, then both LHS and RHS are both equal to zero because $\binom{n}{k} = \binom{\alpha}{k} = 0$ by definition. $\square$

4. In a partition of the power set $P(S)$ of $S = \{1, 2, \ldots, n\}$ into symmetric chains, find a formula for the number of chains of size 1, size 2, and size k, respectively.

Solution. We claim that the number of symmetric chains of size larger than k is

$$\binom{n}{\lceil (n+k)/2 \rceil}.$$

Consider a symmetric chain

$$A_1 \subset A_2 \subset \cdots \subset A_l$$

of size $l \geq k+1$. Since

$$|A_1| + |A_l| = 2|A_1| + l - 1 = n,$$

we have

$$\begin{aligned} |A_1| &= \frac{n-l+1}{2} \leq \frac{n-k}{2}, \\ |A_l| &= \frac{n+l-1}{2} \geq \frac{n+k}{2}. \end{aligned}$$

Hence

$$|A_1| \leq \left\lfloor \frac{n-k}{2} \right\rfloor \leq \frac{n-k}{2} \leq \frac{n+k}{2} \leq \left\lceil \frac{n+k}{2} \right\rceil \leq |A_l|.$$

This means that each symmetric chain of length at leat $k+1$ contains exactly one $\lfloor \frac{n-k}{2} \rfloor$-subset and exactly one $\lceil \frac{n+k}{2} \rceil$-subset of S. Conversely, since

$$\frac{n+k}{2} - \frac{n-k}{2} = k,$$

then any symmetric chain that contains one $\lfloor \frac{n-k}{2} \rfloor$-subset and one $\lceil \frac{n+k}{2} \rceil$-subset must contain at leat $k+1$ subsets. We thus conclude that the number of symmetric chains of size larger than k is

$$\binom{n}{\lceil (n+k)/2 \rceil} = \binom{n}{\lfloor (n-k)/2 \rfloor}.$$

It is clear that the number of symmetric chains of size k ($k \geq 1$) is

$$\binom{n}{\lceil (n+k-1)/2 \rceil} - \binom{n}{\lceil (n+k)/2 \rceil}.$$

5. Assume the expansion formula

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \quad |z| < 1.$$

Prove by induction on n the following expansion formula

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k, \quad |z| < 1.$$

Proof. For $n = 1$, it is obviously true because $\binom{n+k-1}{k} = \binom{k}{k} = 1$. For $n \geq 2$, suppose

$$\frac{1}{(1-z)^{n-1}} = \sum_{k=0}^{\infty} \binom{n-1+k-1}{k} z^k, \quad |z| < 1.$$

Then

$$\begin{aligned} \frac{1}{(1-z)^n} &= \frac{1}{1-z} \cdot \frac{1}{(1-z)^{n-1}} \\ &= \left(\sum_{i=0}^{\infty} z^i\right) \left[\sum_{j=0}^{\infty} \binom{n-1+j-1}{j} z^j\right] \\ &= \sum_{k=0}^{\infty} \left[\sum_{\substack{i+j=k \\ i\geq 0, j\geq 0}} \binom{n+j-2}{j}\right] z^k. \end{aligned}$$

Note that for $k \geq 0$ and $l \geq 1$,

$$\binom{l+k}{l} = \binom{l-1}{l-1} + \binom{l}{l-1} + \cdots + \binom{l-1+k}{l-1}.$$

We thus have

$$\begin{aligned} \sum_{i+j=k} \binom{n+j-2}{j} &= \sum_{j=0}^{k} \binom{n+j-2}{j} = \sum_{j=0}^{k} \binom{n-2+j}{n-2} \\ &= \binom{n-2}{n-2} + \binom{n-1}{n-2} + \cdots + \binom{n-2+k}{n-2} \\ &= \binom{n-1+k}{n-1} = \binom{n+k-1}{k}. \end{aligned}$$

□

6. Consider the partially ordered set $\{1, 2, \ldots, 12\}$ whose partial order is the divisibility.

   (a) Determine a chain of largest size, and a partition of $\{1, 2, \ldots, 12\}$ into the smallest number of antichains.

   (b) Determine an antichain of largest size, and a partition of $\{1, 2, \ldots, 12\}$ into the smallest number of chains.

(a) An antichain partition with four antichains: $\{1\}$, $\{2, 3, 5, 7, 11\}$, $\{4, 6, 10\}$, $\{8, 9, 12\}$.

There is one chain of length four: $\{1, 2, 4, 8\}$.

(b) A chain partition with six chains: $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, $\{11\}$.

There are several antichains of largest size. For instance, $\{2, 6, 5, 7, 9, 11\}$, $\{4, 6, 5, 7, 9, 11\}$, $\{4, 6, 7, 9, 11, 10\}$.

7. Determine the number of 12-combinations of the multiset $\{4a, 3b, 4c, 5d\}$.

Solution. Let S be the set of permutations of the multiset $M = \{\infty a, \infty b, \infty c, \infty d\}$. A_1, A_2, A_3, A_4 be the sets of permutations of M such that the number of a's are more than 4, the number of b's are more than 3, the number of c's are more than 4, and the number of d's are more than 5, respectively. Then

$$|S| = \left\langle {4 \atop 16} \right\rangle = \binom{4+16-1}{16} = \binom{19}{16};$$

$$|A_1| = |A_3| = \left\langle {4 \atop 11} \right\rangle = \binom{4+11-1}{11} = \binom{14}{11},$$

$$|A_2| = \left\langle {4 \atop 12} \right\rangle = \binom{4+12-1}{12} = \binom{15}{12},$$

$$|A_4| = \left\langle {4 \atop 10} \right\rangle = \binom{4+10-1}{10} = \binom{13}{10};$$

$$|A_1 \cap A_2| = |A_2 \cap A_3| = \left\langle {4 \atop 7} \right\rangle = \binom{4+7-1}{7} = \binom{10}{7},$$

$$|A_1 \cap A_4| = |A_3 \cap A_4| = \left\langle {4 \atop 5} \right\rangle = \binom{4+5-1}{5} = \binom{8}{5},$$

$$|A_1 \cap A_3| = |A_2 \cap A_4| = \left\langle {4 \atop 6} \right\rangle = \binom{4+6-1}{6} = \binom{9}{6};$$

$$|A_1 \cap A_2 \cap A_3| = \left\langle {4 \atop 2} \right\rangle = \binom{4+2-1}{2} = \binom{5}{2} = 10,$$

$$|A_1 \cap A_2 \cap A_4| = |A_2 \cap A_3 \cap A_4| = \left\langle {4 \atop 1} \right\rangle = 4,$$

$$|A_1 \cap A_3 \cap A_4| = \left\langle {4 \atop 0} \right\rangle = 1;$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$$

By the inclusion-exclusion formula, the answer is given by

$$\binom{19}{16} - \left[2\binom{14}{11} + \binom{15}{12} + \binom{13}{10}\right] + 2\left[\binom{10}{7} + \binom{8}{5} + \binom{9}{6}\right] - (10 + 2\cdot 4 + 1) + 0.$$

8. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

Solution. Let S be the set of all permutations of $\{1, 2, \ldots, 8\}$. The even integers in $\{1, 2, \ldots, 8\}$ are 2,4,6,8. Let A_1, A_2, A_3, A_4 be the sets of permutations that 2,4,6,8 are fixed respectively. Then

$$|S| = 8!,$$

$$|A_1 = |A_2| = |A_3 = |A_4| = 7!,$$

$$|A_i \cap A_j| = 6!, \quad (1 \le i < j \le 8),$$

$$|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = 5!,$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 4!.$$

Thus by the inclusion-exclusion formulas, the answer is given by

$$8! - 4 \times 7! + 6 \times 6! - 4 \times 5! + 4!.$$

9. Using combinatorial reasoning to prove the identity

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k} = \sum_{k=0}^{n} \binom{n}{k} D_k.$$

Proof. Let S be the set of all permutations of $\{1, 2, \ldots, n\}$. Let A_k be the set of all permutations that k integers are fixed at their positions. Then $|S| = n!$ and $|A_k| = \binom{n}{k} D_{n-k}$. The identity follows from the disjoint union $S = \bigcup_{k=0}^{n} A_k$. $\square$

10. What is the number of ways to place six non-attacking rooks on the 6-by-6 boards with forbidden positions as shown?

(a)

×	×				
		×	×		
				×	×

(b)

×	×				
×	×				
		×	×		
		×	×		
				×	×
				×	×

(c)

×	×				
	×	×			
		×			
				×	×
					×

Recall that the number $R_n(C)$ of ways to place n non-attacking rooks on the n-by-n board C with forbidden positions is given by

$$R_n(C) = \sum_{k=0}^{n} (-1)^k r_k(C)(n-k)!,$$

where $r_k(C)$ is the number of ways to place k non-attacking rooks on the board C. In all three cases, $n = 6$.

(a) Since $r_0 = 1$, $r_1 = 6$, $r_2 = 3 \times 2 \times 2 = 12$, $r_3 = 2 \times 2 \times 2 = 8$, $r_4 = r_5 = r_6 = 0$, then

$$R_6(C) = 6! - 6 \times 5! + 12 \times 4! - 8 \times 3!.$$

(b) Since the rook polynomial

$$\begin{aligned} R(C, x) &= \left(1 + 4x + 2x^2\right)^3 \\ &= \left(1 + 8x + 20x^2 + 16x^3 + 4x^4\right)\left(1 + 4x + 2x^2\right) \\ &= 1 + 12x + 54x^2 + 102x^3 + 44x^4 + 48x^5 + 8x^6, \end{aligned}$$

then $r_0 = 1$, $r_1 = 12$, $r_2 = 54$, $r_3 = 102$, $r_4 = 44$, $r_5 = 48$, and $r_6 = 8$. Thus

$$R_6(C) = 6! - 12 \times 5! + 54 \times 4! - 102 \times 3! + 44 \times 2! - 48 \times 1! + 8 \times 0!.$$

(c) Since the rook polynomial

$$R(C,x) = \left(1+5x+6x^2+x^3\right)\left(1+3x+x^2\right) = 1+8x+22x^2+24x^3+9x^4+x^5,$$

then $r_0 = 1$, $r_1 = 8$, $r_2 = 22$, $r_3 = 24$, $r_4 = 9$, $r_5 = 1$, $r_6 = 0$. Thus

$$R_6(C) = 6! - 8\times 5! + 22\times 4! - 24\times 3! + 9\times 2! - 1!.$$

11. How many circular permutations are there of the multiset

$$\{2a, 3b, 4c, 5d\}$$

so that the elements of the same type are not all consecutively together?

Solution. Let S be the set of all circular permutations of $M = \{2a, 3b, 4c, 5d\}$. Then

$$|S| = \frac{13!}{2!3!4!5!}.$$

Let A_1, A_2, A_3, and A_4 be the sets of circular permutations that the type a, the type b, the type c, and the type d elements are consecutively together respectively. Then

$$|A_1| = \frac{12!}{3!4!5!},\quad |A_2| = \frac{11!}{2!4!5!},\quad |A_3| = \frac{10!}{2!3!5!},\quad |A_4| = \frac{9!}{2!3!4!};$$

$$|A_1\cap A_2| = \frac{10!}{4!5!},\quad |A_1\cap A_3| = \frac{9!}{3!5!},\quad |A_1\cap A_4| = \frac{8!}{3!4!},$$

$$|A_2\cap A_3| = \frac{8!}{2!5!},\quad |A_2\cap A_4| = \frac{7!}{2!4!},\quad |A_3\cap A_4| = \frac{6!}{2!3!};$$

$$|A_1\cap A_2\cap A_3| = \frac{7!}{5!},\quad |A_1\cap A_2\cap A_4| = \frac{6!}{4!},\quad |A_1\cap A_3\cap A_4| = \frac{5!}{3!},\quad |A_2\cap A_3\cap A_4| = \frac{4!}{2!};$$

$$|A_1\cap A_2\cap A_3\cap A_4| = 3!.$$

Thus the answer is given by

$$\frac{13!}{2!3!4!5!} - \left(\frac{12!}{3!4!5!} + \frac{11!}{2!4!5!} + \frac{10!}{2!3!5!} + \frac{9!}{2!3!4!}\right)$$

$$+\left(\frac{10!}{4!5!} + \frac{9!}{3!5!} + \frac{8!}{3!4!} + \frac{8!}{2!5!} + \frac{7!}{2!4!} + \frac{6!}{2!3!}\right) - \left(\frac{7!}{5!} + \frac{6!}{4!} + \frac{5!}{3!} + \frac{4!}{2!}\right) + 3!.$$