Week 3-4: Permutations and Combinations

March 10, 2020

1 Two Counting Principles

Addition Principle. Let $S_1, S_2, \ldots, S_m$ be disjoint subsets of a finite set S. If $S = S_1 \cup S_2 \cup \cdots \cup S_m$, then

$$|S| = |S_1| + |S_2| + \cdots + |S_m|.$$

Multiplication Principle. Let $S_1, S_2, \ldots, S_m$ be finite sets and $S = S_1 \times S_2 \times \cdots \times S_m$. Then

$$|S| = |S_1| \times |S_2| \times \cdots \times |S_m|.$$

Example 1.1. Determine the number of positive integers which are factors of the number $5^3 \times 7^9 \times 13 \times 33^8$.

The number 33 can be factored into 3×11. By the unique factorization theorem of positive integers, each factor of the given number is of the form $3^i \times 5^j \times 7^k \times 11^l \times 13^m$, where $0 \le i \le 8$, $0 \le j \le 3$, $0 \le k \le 9$, $0 \le l \le 8$, and $0 \le m \le 1$. Thus the number of factors is

$$9 \times 4 \times 10 \times 9 \times 2 = 7280.$$

Example 1.2. How many two-digit numbers have distinct and nonzero digits?

A two-digit number ab can be regarded as an ordered pair (a, b) where a is the tens digit and b is the units digit. The digits in the problem are required to satisfy

$$a \neq 0, \quad b \neq 0, \quad a \neq b.$$

The digit a has 9 choices, and for each fixed a the digit b has 8 choices. So the answer is $9 \times 8 = 72$.

The answer can be obtained in another way: There are 90 two-digit numbers, i.e., $10, 11, 12, \ldots, 99$. However, the 9 numbers $10, 20, \ldots, 90$ should be excluded; another 9 numbers $11, 22, \ldots, 99$ should be also excluded. So the answer is $90 - 9 - 9 = 72$.

Example 1.3. How many odd numbers between 1000 and 9999 have distinct digits?

A number $a_1a_2a_3a_4$ between 1000 and 9999 can be viewed as an ordered tuple (a_1, a_2, a_3, a_4). Since $a_1a_2a_3a_4 \geq 1000$ and $a_1a_2a_3a_4$ is odd, then $a_1 = 1, 2, \ldots, 9$ and $a_4 = 1, 3, 5, 7, 9$. Since a_1, a_2, a_3, a_4 are distinct, we conclude: a_4 has 5 choices; when a_4 is fixed, a_1 has 8 $(= 9 - 1)$ choices; when a_1 and a_4 are fixed, a_2 has 8 $(= 10 - 2)$ choices; and when a_1, a_2, a_4 are fixed, a_3 has 7 $(= 10 - 3)$ choices. Thus the answer is $8 \times 8 \times 7 \times 5 = 2240$.

Example 1.4. In how many ways to make a basket of fruit from 6 oranges, 7 apples, and 8 bananas so that the basket contains at least two apples and one banana?

Let a_1, a_2, a_3 be the number of oranges, apples, and bananas in the basket respectively. Then $0 \leq a_1 \leq 6$, $2 \leq a_2 \leq 7$, and $1 \leq a_3 \leq 8$, i.e., a_1 has 7 choices, a_2 has 6 choices, and a_3 has 8 choices. Thus the answer is $7 \times 6 \times 8 = 336$.

General Ideas about Counting:

- Count the number of **ordered** arrangements or **ordered** selections of objects

  (a) without repetition,

  (b) with repetition allowed.

- Count the number of **unordered** arrangements or **unordered** selections of objects

  (a) without repetition,

(b) with repetition allowed.

A **multiset** M is a collection whose members need not be distinct. For instance, the collection

$$M = \{a, a, b, b, c, d, d, d, 1, 2, 2, 2, 3, 3, 3, 3\}$$

is a multiset; and sometimes it is convenient to write

$$M = \{2a, 2b, c, 3d, 1, 3 \cdot 2, 4 \cdot 3\}.$$

A multiset M over a set S can be viewed as a function $v : S \to \mathbb{N}$ from S to the set $\mathbb{N}$ of nonnegative integers; each element $x \in S$ is repeated $v(x)$ times in M; we write $M = (S, v)$.

Example 1.5. How many integers between 0 and 10,000 have exactly one digit equal to 5?

First Method. Let S be the set of such numbers, and let S_i be the set of such numbers having exactly i digits, $1 \leq i \leq 4$. Clearly, $|S_1| = 1$. For S_2, if the tens is 5, then the units has 9 choices; if the units is 5, then the tens has 8 choices; thus $|S_2| = 9 + 8 = 17$.

For S_3, if the tens is 5, then the units has 9 choices and the hundreds has 8 choices; if the hundreds is 5, then both tens and the units have 9 choices; if the units is 5, then the tens has 9 choices and hundreds has 8 choices; thus $|S_3| = 9 \times 9 + 8 \times 9 + 8 \times 9 = 225$.

For S_4, if the thousands is 5, then each of the other three digits has 9 choices; if the hundreds or tens or units is 5, then the thousands has 8 choices, each of the other two digits has 9 choices; thus $|S_4| = 9 \times 9 \times 9 + 3 \times 8 \times 9 \times 9 = 2,673$.

Therefore

$$|S| = |S_1| + |S_2| + |S_3| + |S_4| = 1 + 17 + 225 + 2,673 = 2,916.$$

Second Method. Let us write any integer with less than 5 digits in a formal 5-digit form by adding zeros in the front. For instance, we write 35 as 00035, 836 as 00836. Let S_i be the set of integers of S whose ith digit is 5, $1 \leq i \leq 4$. Then $|S_i| = 9 \times 9 \times 9 = 729$. Thus $|S| = 4 \times 729 = 2,916$.

Example 1.6. How many distinct 5-digit numerals can be constructed out of the digits $1, 1, 1, 6, 8$?

The digit 6 can be located in any of the 5 positions; then 8 can be located in 4 positions. Thus the answer is $5 \times 4 = 20$.

2 Permutation of Sets

Definition 2.1. An r**-permutation** of n objects is a linearly ordered selection of r objects from a set of n objects. The number of r-permutations of n objects is denoted by

$$P(n, r).$$

An n-permutation of n objects is just called a **permutation of** n **objects**. The number of permutations of n objects is denoted by $n!$, read "n factorial."

Theorem 2.2. *The number of* r*-permutations of an* n*-set equals*

$$P(n, r) = n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}.$$

Corollary 2.3. *The number of permutations of an* n*-set is* $n!$.

Example 2.1. Find the number of ways to put the numbers $1, 2, \ldots, 8$ into the squares of 6-by-6 grid so that each square contains at most one number.

There are 36 squares in the 6-by-6 grid below. We label the squares by the numbers $1, 2, ..., 36$ as follows:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

		5			
3				7	
			4		
		6	2		
	8				
				1	

The filling pattern on the right can be viewed as an 8-permutation $(35, 22, 7, 16, 3, 21, 11, 26)$ of $\{1, 2, \ldots, 36\}$. Thus the answer is

$$P(36, 8) = \frac{36!}{(36-8)!} = \frac{36!}{28!}.$$

Example 2.2. What is the number of ways to arrange the 26 alphabets so that no two of the vowels a, e, i, o, and u occur next to each other?

We first have the 21 consonants arranged arbitrarily and there are 21! ways to do so. For each such 21-permutation, we arrange the 5 vowels a, e, i, o, u in 22 positions between consonants; there are $P(22, 5)$ ways of such arrangement. Thus the answer is

$$21! \times P(22, 5) = 21! \times \frac{22!}{17!}.$$

Example 2.3. Find the number of 7-digit numbers in base 10 such that all digits are nonzero, distinct, and the digits 8 and 9 do not appear next to each other.

First Method. The numbers in question can be viewed as 7-permutations of $\{1, 2, \ldots, 9\}$ with certain restrictions. Such permutations can be divided into three types: (i) permutations without 8 and 9; (ii) permutations with either 8 or 9 but not both; and (iii) permutations with both 8 and 9, but not next to each other.

(i) There are $P(7, 7) = 7! = 5,040$ such permutations.

(ii) There are $P(7, 6)$ 6-permutations of $\{1, 2, \ldots, 7\}$. Thus there are $2 \times 7 \times P(7, 6) = 2 \times 7 \times \frac{7!}{1!} = 70,560$ such permutations.

(iii) For each 5-permutation of $\{1, 2, \ldots, 7\}$, there are 6 ways to insert 8 in it, and then there are 5 ways to insert 9. Thus there are $6 \times 5 \times P(7, 5) = 75,600$.

Therefore the answer is

$$5,040 + 70,560 + 75,600 = 151,200.$$

Second Method. Let S be the set of 7-permutations of $\{1, 2, \ldots, 9\}$. Let A be the subset of 7-permutations of S in the problem. Then $\bar{A}$ is the set of 7-permutations of S such that either 89 or 98 appears somewhere. We may think of 89 and 98 as a single object in whole, then $\bar{A}$ can be viewed as the set of 6-permutations of $\{1, 2, 3, 4, 5, 6, 7, 89\}$ with 89 and 6-permutations of $\{1, 2, 3, 4, 5, 6, 7, 98\}$ with 98. It follows that $|\bar{A}| = 2(P(8, 6) - P(7, 6))$. Thus the answer is

$$|A| = P(9, 7) - 2\big(P(8, 6) - P(7, 6)\big) = \frac{9!}{2!} - 2\left(\frac{8!}{2!} - \frac{7!}{1!}\right) = 151,200.$$

The set $\bar{A}$ can be obtained by taking all 5-permutations of $\{1, 2, \ldots, 7\}$ first and then by adding 89 or 98 to one of 6 positions of the 5-permutations. Then

$$|A| = P(9, 7) - 2P(7, 5) \cdot 6 = \frac{9!}{2!} - 7! \times 6 = 151, 200.$$

A **circular r-permutation** of a set S is an ordered r objects of S arranged as a circle; there is no the beginning object and the ending object.

Theorem 2.4. *The number of circular r-permutations of an n-set equals*

$$\frac{P(n, r)}{r} = \frac{n!}{(n-r)!r}.$$

Proof. Let S be an n-set. Let X be the set of all r-permutations of S, and let Y be the set of all circular r-permutations of S. Define a function

$$f : X \to Y, \quad a_1a_2 \cdots a_r \mapsto \ldots a_ra_1a_2 \ldots a_ra_1a_2 \ldots$$

as follows: For each r-permutation $a_1a_2 \cdots a_r$ of S, $f(a_1a_2 \cdots a_r)$ is the circular r-permutation such that $a_1a_2 \ldots a_ra_1a_2 \ldots$ is counterclockwise on a circle. Clearly, f is surjective. Moreover, there are exactly r-permutations sent to one circular r-permutation. In fact, the r permutations

$$a_1a_2a_3 \ldots a_{r-1}a_r, \quad a_2a_3a_4 \ldots a_ra_1, \quad \ldots, \quad a_ra_1a_2 \ldots a_{r-2}a_{r-1}$$

are sent to the same circular r-permutation. Thus the answer is

$$|Y| = \frac{|X|}{r} = \frac{P(n, r)}{r}.$$

□

Corollary 2.5. *The number of circular permutations of an n-set is*

$$(n-1)!.$$

Example 2.4. Twelve people, including two who do no wish to sit next to each other, are to be seated at a round table. How many circular seating plans can be made?

First Method. We may have 11 people (including one of the two unhappy persons but not both) to sit first; there are 10! such seating plans. Next the second unhappy person can sit anywhere except the left side and right side of the first unhappy person; there are 9 choices for the second unhappy person. Thus the answer is $9 \times 10!$.

Second Method. There are 11! seating plans for the 12 people with no restriction. We need to exclude those seating plans that the unhappy persons a and b sit next to each other. Note that a and b can sit next to each other in two ways: ab and ba. We may view a and b as an inseparable twin; there are $2 \times 10!$ such seating plans. Thus the answer is given by

$$11! - 2 \times 10! = 9 \times 10!.$$

Example 2.5. How many different patterns of necklaces with 18 beads can be made out of 25 available beads of the same size but in different colors?

Answer: $\frac{P(25,18)}{18 \times 2} = \frac{25!}{36 \times 7!}$.

3 Combinations of Sets

A **combination** is a collection of objects (order is immaterial) from a given set. An r-**combination** of an n-set S is an r-subset of S. We denote by $\binom{n}{r}$ the number of r-combinations of an n-set, read "n choose r."

Theorem 3.1. *The number of r-combinations of an n-set equals*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{P(n,r)}{r!}.$$

First Proof. Let S be an n-set. Let X be the set of all permutations of S, and let Y be the set of all r-subsets of S. Consider a map $f : X \to Y$ defined by

$$f(a_1 a_2 \ldots a_r a_{r+1} \ldots a_n) = \{a_1, a_2, \ldots, a_r\}, \quad a_1 a_2 \ldots a_n \in X.$$

Clearly, f is surjective. Moreover, for each r-subset $A = \{a_1, a_2, \ldots, a_r\} \in Y$, there are $r!$ permutations of A and $(n-r)!$ permutations of the complement $\bar{A}$. Then

$$f^{-1}(A) = \{\sigma\tau : \sigma \text{ is a permutation of } A \text{ and } \tau \text{ is a permutation of } \bar{A}\}.$$

Thus $|f^{-1}(A)| = r!(n-r)!$. Therefore

$$\binom{n}{r} = |Y| = \frac{|X|}{r!(n-r)!} = \frac{n!}{r!(n-r)!}.$$

Second Proof. Let X be the set of all r-permutations of S, and Y the set of all r-subsets of S. Consider a map $f : X \to Y$ defined by

$$f(a_1 a_2 \ldots a_r) = \{a_1, a_2, \ldots, a_r\}, \quad a_1 a_2 \ldots a_r \in X.$$

Clearly, f is surjective. Moreover, there are exactly $r!$ permutations of $\{a_1, a_2, \ldots, a_r\}$ sent to $\{a_1, a_2, \ldots, a_r\}$. Thus

$$\binom{n}{r} = |Y| = \frac{|X|}{r!} = \frac{P(n,r)}{r!}.$$

$\square$

Example 3.1. How many 8-letter words can be constructed from 26 letters of the alphabets if each word contains 3, 4, or 5 vowels? It is understood that there is no restriction on the number of times a letter can be used in a word.

The number of words with 3 vowels: There are $\binom{8}{3}$ ways to choose 3 vowel positions in a word; each vowel position can be filled with one of the 5 vowels; the consonant position can be any of 21 consonants. Thus there are $\binom{8}{3} 5^3 21^5$ words having exactly 3 vowels.

The number of words with 4 vowels: $\binom{8}{4} 5^4 21^4$.

The number of words with 5 vowels: $\binom{8}{5} 5^5 21^3$.

Thus the answer is

$$\binom{8}{3} 5^3 21^5 + \binom{8}{4} 5^4 21^4 + \binom{8}{5} 5^5 21^3.$$

Corollary 3.2. *For integers n, r such that $n \geq r \geq 0$,*

$$\binom{n}{r} = \binom{n}{n-r}.$$

Theorem 3.3. *The number of subsets of an n-set S equals*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

4 Permutations of Multisets

Let M be a multiset. An r**-permutation** of M is an ordered arrangement of r objects of M. If $|M| = n$, then an n-permutation of M is called a **permutation** of M.

Theorem 4.1. *Let M be a multiset of k different types where each type has infinitely many elements. Then the number of r-permutations of M equals*

$$k^r.$$

Example 4.1. What is the number of ternary (base-3) numerals with at most 4 digits?

The question is to find the number of 4-permutations of the multiset $\{\infty 0, \infty 1, \infty 2\}$. Thus the answer is $3^4 = 81$.

Theorem 4.2. *Let M be a multiset of k types with repetition numbers $n_1, n_2, \ldots, n_k$ respectively. Let $n = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of M equals*

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

Proof. List the elements of M as

$$\underbrace{a, \ldots, a}_{n_1}, \underbrace{b, \ldots, b}_{n_2}, \ldots, \underbrace{d, \ldots, d}_{n_k}.$$

Let S be the set consisting of the elements $a_1, a_2, \ldots, a_{n_1}$, $b_1, b_2, \ldots, b_{n_2}$, ..., $d_1, d_2, \ldots, d_{n_k}$. Let X be the set of all permutations of S, and let Y be the set of all permutations of M. There is a map $f : X \to Y$, sending each permutation of S to a permutation of M by removing the subscripts of the elements. Note that for each permutation π of M there are $n_1!$, $n_2!$, ..., and $n_k!$ ways to put the subscripts of the first, the second, ..., and the kth type elements back, independently. Thus there are $n_1! n_2! \cdots n_k!$ elements of X sent to π by f. Therefore the answer is

$$|Y| = \frac{|X|}{n_1! n_2! \cdots n_k!} = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

□

Corollary 4.3. *The number of* 0-1 *words of length n with exactly r ones and $n-r$ zeros is equal to*

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Example 4.2. How many possibilities are there for 8 non-attacking rooks on an 8-by-8 chessboard? How about having 8 different color rooks? How about having 1 red (R) rook, 3 blue (B) rooks , and 4 yellow (Y) rooks.

We label each square by an ordered pair (i,j) of coordinates, $(1,1) \leq (i,j) \leq (8,8)$. Then the rooks must occupy 8 squares with coordinates

$$(1,a_1), \quad (2,a_2), \quad \ldots, \quad (8,a_8),$$

where $a_1, a_2, \ldots, a_8$ must be distinct, i.e., $a_1a_2\ldots a_8$ is a permutation of $\{1,2,\ldots,8\}$. Thus the answer is 8!.

When the 8 rooks have different colors, the answer is $8!8! = (8!)^2$.

If there are 1 red rook, 2 blue rooks, and 3 yellow rooks, then we have a multiset $M = \{R, 3B, 4Y\}$ of rooks. The number of permutations of M equals $\frac{8!}{1!3!4!}$, and the answer in question is $8! \times \frac{8!}{1!3!4!}$.

Theorem 4.4. *Given n rooks of k colors with n_1 rooks of the first color, n_2 rooks of the second color, ..., and n_k rooks of the kth color. The number of ways to arrange these rooks on an n-by-n board so that no one can attack another equals*

$$n! \times \frac{n!}{n_1!n_2!\cdots n_k!} = \frac{(n!)^2}{n_1!n_2!\cdots n_k!}.$$

Example 4.3. Find the number of 8-permutations of the multiset

$$M = \{a,a,a,b,b,c,c,c,c\} = \{3a,2b,4c\}.$$

The number of 8-permutations of $\{2a,2b,4c\}$: $\frac{8!}{2!2!4!}$.
The the number of 8-permutations of $\{3a,b,4c\}$: $\frac{8!}{3!1!4!}$.
The number of 8-permutations of $\{3a,2b,3c\}$: $\frac{8!}{3!2!3!}$.
Thus the answer is

$$\frac{8!}{2!2!4!} + \frac{8!}{3!1!4!} + \frac{8!}{3!2!3!} = 420 + 280 + 560 = 1,260.$$

5 Combinations of Multisets

Let M be a multiset. An **r-combination** of M is an unordered collection of r objects from M. Thus an r-combination of M is itself an r-submultiset of M. For a multiset $M = \{\infty a_1, \infty a_2, \ldots, \infty a_n\}$, an r-combination of M is also called an **r-combination with repetition allowed** of the n-set $S = \{a_1, a_2, \ldots, a_n\}$. The number of r-combinations with repetition allowed of an n-set is denoted by

$$\left\langle {n \atop r} \right\rangle.$$

Theorem 5.1. *Let $M = \{\infty a_1, \infty a_2, \ldots, \infty a_n\}$ be a multiset of n types. Then the number of r-combinations of M is given by*

$$\left\langle {n \atop r} \right\rangle = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

Proof. When r objects are taken from the multiset M, we put them into the following boxes

$$\begin{array}{c|c|c|c} \quad & \quad & \quad & \quad \\ \hline \text{1st} & \text{2nd} & \cdots \quad \cdots \quad \cdots & n\text{th} \\ \hline \end{array}$$

so that the ith type objects are contained in the ith box, $1 \leq i \leq n$. Since the objects of the same type are indistinguishable, we may use the symbol "O" to denote an object in the boxes, and the objects in different boxes are separated by a stick "|". Convert the symbol "O" to zero 0 and the stick "|" to one 1, any such placement is converted into a 0-1 sequence of length $r + n - 1$ with exactly r zeros and $n - 1$ ones. For example, for $n = 4$ and $r = 7$,

$$\{a, a, b, c, c, c, d\} \longleftrightarrow \begin{array}{c|c|c|c} \text{OO} & \text{O} & \text{OOO} & \text{O} \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \longleftrightarrow 0010100010$$

$$\{b, b, b, b, d, d, d\} \longleftrightarrow \begin{array}{c|c|c|c} & \text{OOOO} & & \text{OOO} \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \longleftrightarrow 1000011000$$

Now the problem becomes counting the number of 0-1 words of length $r+(n-1)$ with exactly r zeros and $n - 1$ ones. Thus the answer is

$$\left\langle {n \atop r} \right\rangle = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

$\square$

Corollary 5.2. *The number* $\left\langle {n \atop r} \right\rangle$ *equals the number of ways to place* r *identical balls into* n *distinct boxes.*

Corollary 5.3. *The number* $\left\langle {n \atop r} \right\rangle$ *equals the number of nonnegative integer solutions of the equation*

$$x_1 + x_2 + \cdots + x_n = r.$$

Corollary 5.4. *The number* $\left\langle {n \atop r} \right\rangle$ *equals the number of nondecreasing sequences of length* r *whose terms are taken from the set* $\{1, 2, \ldots, n\}$*.*

Proof. Each nondecreasing sequence $a_1 \leq a_2 \leq \cdots \leq a_r$ with $1 \leq a_i \leq n$ can be identified as an r-combination $\{a_1, a_2, \ldots, a_r\}$ (an r-multiset) from the n-set $\{1, 2, \ldots, n\}$ with repetition allowed, and vice versa. $\square$

Example 5.1. Find the number of nonnegative integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 < 19.$$

The problem is equivalent to finding the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 18$$

So the answer is $\left\langle {5 \atop 18} \right\rangle = \binom{22}{18} = \binom{22}{4} = 43890.$